A Minimal Contractor for the Polar Equation; Application to Robot Localization

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Abstract

Contractor programming relies on a catalog on elementary contractors which need to be as efficient as possible. In this paper, we introduce a new theorem that can be used to build minimal contractors consistent with equations, and another new theorem to derive an optimal separator from a minimal contractor. As an application, we focus on the channeling polar constraint associated to the change between Cartesian coordinates and Polar coordinates. We illustrate our method on the localization problem of an actual underwater robot where both range and goniometric measurements of landmarks are collected.
Keywords: Set theory, Interval Analysis, Constraint programming, Localization, Robotics

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1. Introduction

Contractor programming [7] is an efficient tool to solve rigorously complex problems involving uncertainties and nonlinear equations [6, 22]. A *contractor* $\mathcal{C}_{\mathbb{X}}$ is an operator able to contract a box of $\mathbb{R}^n$ without removing a single point of the subset $\mathbb{X}$ of $\mathbb{R}^n$ to which it is associated. As a result, using a paving of $\mathbb{R}^n$ generated by a paver [25], the contractor will allow us to build an outer approximation of $\mathbb{S}$. Basic notions on interval analysis, contractors and applications can be found in [19].

Contractor programming relies on a catalog of elementary contractors. Most of the time, these elementary contractors are built using interval arithmetic [24]. Then, by

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combining all these elementary contractors, we can construct a more sophisticated contractor consistent with the solution set of the problem we want to solve. The principle can be extended to separator programming [18] in order to compute an inner and an outer approximation of the solution set.

Now, combining contractors introduces a pessimism which has to be balanced by additional bisections performed by the paver. For more efficiency, it is important to extend the catalog by adding some new specific contractors.

In this paper, we propose some new theorems in order to build more easily optimal contractors/separators consistent with equations often used, for instance, in the field of robotics [21, 10, 23]. As an application, we will consider the *polar* constraint associated to the change of coordinates between Cartesian and polar form [4]. This polar constraint is essential for localization of robots when both goniometric and/or distance measurements are available [9, 14]. Some test cases will show that our approach makes it possible to obtain an inner and an outer approximation of the solution set in a much more efficient manner than simply composing elementary interval contractors.

This paper is organized as follows. Section 2 presents the notion of contractor and separator algebra. Section 3 shows how a minimal contractor for some specific constraints can be built. Section 4 will then derive an optimal separator for the polar transformation. Section 5 provides an application on the localization of an actual underwater robot and Section 6 concludes the paper.

2. Contractors and Separators

This section recalls the basic notions on intervals, contractors and separators that are needed to understand the contribution of this paper. An *interval* of $\mathbb{R}$ is a closed connected set of $\mathbb{R}$. A box $[\mathbf{x}]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. Notation used in this paper are given in Table 1.

Subsets of $\mathbb{R}^n$:	$\mathbb{X}, \mathbb{Y}$
Intervals of $\mathbb{R}$:	$[a]$
Boxes of $\mathbb{R}^n$:	$[\mathbf{a}]$
Set of boxes of $\mathbb{R}^n$:	$\mathbb{IR}^n$
Box hull of a set $\mathbb{A}$:	$[\![\mathbb{A}]\!]$
Union hull of two boxes:	$[\mathbf{x}] \sqcup [\mathbf{y}] = [\![[\mathbf{x}] \cup [\mathbf{y}]]\!]$
Composition of functions:	$\mathbf{f} \circ \mathbf{f} = \mathbf{f}^2$

Table 1: Notations

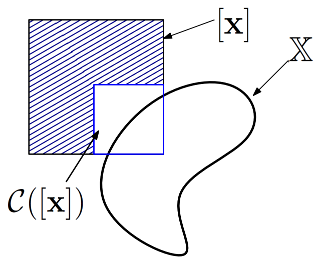


Figure 1: Contractor consistent with to the set $\mathbb{X}$

2.1. Contractors and Separators

In this section, we recall the basic notions on contractors and separators that will be used later. A *contractor* $\mathcal{C}$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ (see e.g., [13]) such that

$$\begin{array}{ll} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ [\mathbf{x}] \subset [\mathbf{y}] \;\Rightarrow\; \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}]). & \text{(monotonicity)} \end{array} \tag{1}$$

We define the inclusion between two contractors $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows:

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall\, [\mathbf{x}] \in \mathbb{IR}^n,\, \mathcal{C}_1([\mathbf{x}]) \subset \mathcal{C}_2([\mathbf{x}]). \tag{2}$$

A set $\mathbb{X}$ is *consistent* (See Figure 1) with the contractor $\mathcal{C}$ (we will write $\mathbb{X} \sim \mathcal{C}$) if for all $[\mathbf{x}]$, we have

$$\mathcal{C}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X}. \tag{3}$$

Two contractors $\mathcal{C}$ and $\mathcal{C}_1$ are equivalent (we will write $\mathcal{C} \sim \mathcal{C}_1$) if we have:

$$\mathbb{X} \sim \mathcal{C} \Leftrightarrow \mathbb{X} \sim \mathcal{C}_1. \tag{4}$$

A contractor $\mathcal{C}$ is *minimal* if for any other contractor $\mathcal{C}_1$, we have the following implication

$$\mathcal{C} \sim \mathcal{C}_1 \Rightarrow \mathcal{C} \subset \mathcal{C}_1. \tag{5}$$

If $\mathcal{C}$ is a minimal contractor consistent with $\mathbb{X}$, then for all $[\mathbf{x}]$, we have $\mathcal{C}([\mathbf{x}]) \cap \mathbb{X} = [\![[\mathbf{x}] \cap \mathbb{X}]\!]$ where $[\![\mathbb{A}]\!]$ is the *hull* operator, *i.e.*, the smallest box which encloses $\mathbb{A}$. This means that $\mathcal{C}([\mathbf{x}])$ corresponds exactly to the smallest box that can be obtained by a contraction of $[\mathbf{x}]$ without removing a single point of $\mathbb{X}$. As a consequence, there exists a unique minimal contractor.

Example 1. The minimal contractor $\mathcal{C}_{\mathbb{X}}$ consistent with the set

$$\mathbb{X} = \left\{ \mathbf{x} \in \mathbb{R}^2, (x_1 - 2)^2 + (x_2 - 2.5)^2 \in [1, 4] \right\} \tag{6}$$

can be built using a forward-backward constraint propagation **[2]** **[15]**. The contractor $\mathcal{C}_{\mathbb{X}}$ can be used by a paver to obtain an outer approximation for $\mathbb{X}$. This is illustrated by Figure **2** (left) where $\mathcal{C}_{\mathbb{X}}$ removes parts of the space outside $\mathbb{X}$ (painted light-gray). But due to the consistency property (see Equation (3)) $\mathcal{C}_{\mathbb{X}}$ has no effect on boxes included in $\mathbb{X}$. A box partially included in $\mathbb{X}$ can not be eliminated and is bisected, except if its length is larger than an given value ε. The contractor $\mathcal{C}_{\mathbb{X}}$ only provides an outer approximation of $\mathbb{X}$.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two contractors, we define the following operations on contractors **[7]**.

$$\begin{array}{rcl} (\mathcal{C}_1 \cap \mathcal{C}_2)([\mathbf{x}]) & = & \mathcal{C}_1([\mathbf{x}]) \cap \mathcal{C}_2([\mathbf{x}]) \quad (7) \\ (\mathcal{C}_1 \sqcup \mathcal{C}_2)([\mathbf{x}]) & = & \mathcal{C}_1([\mathbf{x}]) \sqcup \mathcal{C}_2([\mathbf{x}]) \quad (8) \\ (\mathcal{C}_1 \circ \mathcal{C}_2)([\mathbf{x}]) & = & \mathcal{C}_1 \left(\mathcal{C}_2([\mathbf{x}]) \right) \quad (9) \end{array}$$

where $\sqcup$ is the *union hull* defined by

$$[\mathbf{x}] \sqcup [\mathbf{y}] = [\![[\mathbf{x}] \cup [\mathbf{y}]]\!]. \tag{10}$$

In order to characterize an inner and outer approximation of the solution set, we introduce the notion of *separator*.

A *separator* $\mathcal{S}$ is a pair of contractors $\left\{ \mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}} \right\}$ such that, for all $[\mathbf{x}] \in \mathbb{IR}^n$, we have

$$\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \mathcal{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}] \qquad \text{(complementarity)}. \tag{11}$$

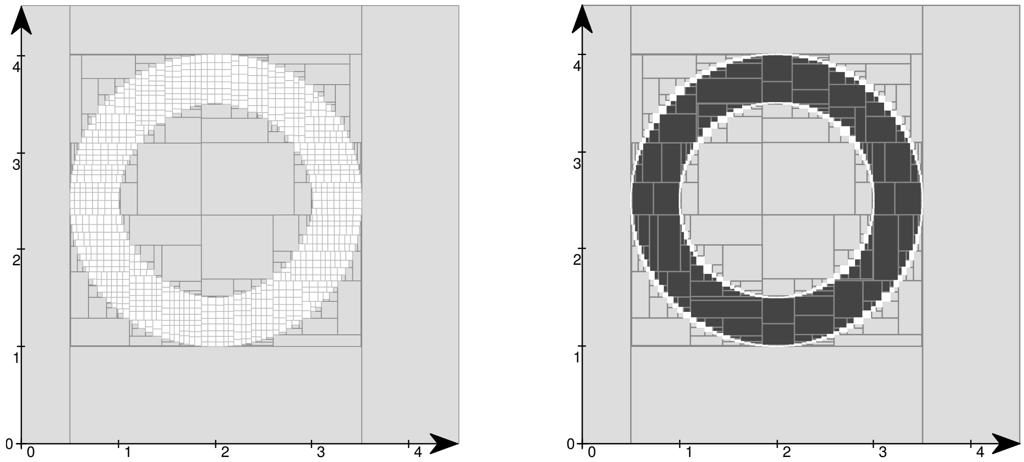


Figure 2: Paving associated to Example 1, Left: paving obtained using the contractor, Right: paving obtained using the separator. Dark gray boxes belong $\mathbb{X}$ (the ring); light gray boxes are outside $\mathbb{X}$. No conclusion can be given on the white boxes.

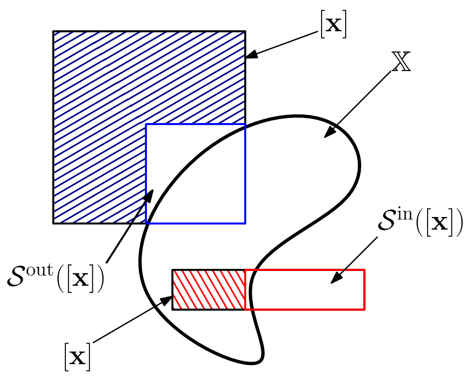


Figure 3: Illustration of a separator on two different initial boxes. The outer contractor removes the blue dashed area and the red dashed area is removed by the inner contractor

A set $\mathbb{X}$ is *consistent* with the separator $\mathcal{S}$ (we will write $\mathbb{X} \sim \mathcal{S}$), if

$$\mathbb{X} \sim \mathcal{S}^{\text{out}} \text{ and } \overline{\mathbb{X}} \sim \mathcal{S}^{\text{in}}, \tag{12}$$

where $\overline{\mathbb{X}} = \{\mathbf{x} \mid \mathbf{x} \notin \mathbb{X}\}$. This notion of separator is illustrated by Figure 3.

We define the inclusion between two separators $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows

$$\mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow \mathcal{S}_1^{\text{in}} \subset \mathcal{S}_2^{\text{in}} \text{ and } \mathcal{S}_1^{\text{out}} \subset \mathcal{S}_2^{\text{out}}. \tag{13}$$

A separator $\mathcal{S}$ is *minimal* if

$$\mathcal{S}_1 \subset \mathcal{S} \Rightarrow \mathcal{S}_1 = \mathcal{S}. \tag{14}$$

It is trivial to check that $\mathcal{S}$ is minimal implies that the two contractors $\mathcal{S}^{\text{in}}$ and $\mathcal{S}^{\text{out}}$ are both minimal. If we define the following operations

$$\begin{array}{rcll} \mathcal{S}_1 \cap \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cup \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cap \mathcal{S}_2^{\text{out}}\right\} & \text{(intersection)} \\ \mathcal{S}_1 \cup \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cap \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cup \mathcal{S}_2^{\text{out}}\right\} & \text{(union)} \end{array} \tag{15}$$

then we have [18]

$$\left\{\begin{array}{ccc} \mathcal{S}_1 & \sim & \mathbb{X}_1 \\ \mathcal{S}_2 & \sim & \mathbb{X}_2 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc} \mathcal{S}_1 \cap \mathcal{S}_2 & \sim & \mathbb{X}_1 \cap \mathbb{X}_2 \\ \mathcal{S}_1 \cup \mathcal{S}_2 & \sim & \mathbb{X}_1 \cup \mathbb{X}_2 \end{array}\right. \tag{16}$$

Example 2. Consider the set $\mathbb{X}$ of Example **1**. From the contractor consistent with

$$\overline{\mathbb{X}} = \left\{\mathbf{x} \in \mathbb{R}^2, (x_1 - 2)^2 + (x_2 - 2.5)^2 \notin [1, 4]\right\}, \tag{17}$$

we can build a separator $\mathcal{S}_{\mathbb{X}}$ for $\mathbb{X}$. An inner and outer approximation of $\mathbb{X}$ obtained by a paver based on $\mathcal{S}_{\mathbb{X}}$ is depicted on Figure 2. The dark gray area is inside $\mathbb{X}$ and light gray is outside. The minimality property of the separators can be observed by the fact that all contracted boxes of the subpaving touch the boundary of $\mathbb{X}$. Therefore, we are now able to quantify the pessimism introduced by the set inversion and to prove the existence of solutions.

2.2. Transformations of Contractors and Separators

We now present some results obtained in [26] and [17] about the symmetries and the minimality of contractors. These results will be used further to get the minimal contractor for the polar equation.

Proposition 1. *If* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are the two minimal contractors consistent with* $\mathbb{X}_1$ *and* $\mathbb{X}_2$ *then* $\mathcal{C}_1 \sqcup \mathcal{C}_2$ *is the minimal contractor consistent with* $\mathbb{X}_1 \cup \mathbb{X}_2$.

Proof. The minimal contractor consistent with $\mathbb{X}_1 \cup \mathbb{X}_2$ is

$$\begin{array}{rcll} [\![(\mathbb{X}_1 \cup \mathbb{X}_2) \cap [\mathbf{x}]]\!] & = & [\![(\mathbb{X}_1 \cap [\mathbf{x}]) \cup (\mathbb{X}_2 \cap [\mathbf{x}])]\!] & (\,(\mathbb{A} \cup \mathbb{B}) \cap \mathbb{C} = (\mathbb{A} \cap \mathbb{C}) \cup (\mathbb{B} \cap \mathbb{C})\,) \\ & = & [\![[\![\mathbb{X}_1 \cap [\mathbf{x}]]\!] \cup [\![\mathbb{X}_2 \cap [\mathbf{x}]]\!]]\!] & ([\![\mathbb{A} \cup \mathbb{B}]\!] = [\![[\![\mathbb{A}]\!] \cup [\![\mathbb{B}]\!]]\!]\ ) \\ & = & [\![\mathbb{X}_1 \cap [\mathbf{x}]]\!] \sqcup [\![\mathbb{X}_2 \cap [\mathbf{x}]]\!] & (\,[\![[\mathbf{x}] \cup [\mathbf{y}]]\!] = [\mathbf{x}] \sqcup [\mathbf{y}]\,) \\ & = & \mathcal{C}_1([\mathbf{x}]) \sqcup \mathcal{C}_2([\mathbf{x}]) & (\text{minimality of } \mathcal{C}_1 \text{ and } \mathcal{C}_2) \end{array}$$

which terminates the proof. ■

Definition 1. A bijective function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is *box-conservative* if for all $\mathbb{A} \subset \mathbb{R}^n$,

$$\mathbf{f}([\![\mathbb{A}]\!]) = [\![\mathbf{f}(\mathbb{A})]\!]. \tag{18}$$

Proposition 2. *If* $\mathbf{f}$ *is box conservative so is* $\mathbf{f}^{-1}$.

Proof.

$$\begin{array}{rcll} \mathbf{f}^{-1}([\![\mathbb{A}]\!]) & = & \mathbf{f}^{-1}\left([\![\mathbf{f} \circ \mathbf{f}^{-1}(\mathbb{A})]\!]\right) & (\mathbf{f} \text{ is bijective}) \\ & = & \mathbf{f}^{-1} \circ \mathbf{f}\left([\![\mathbf{f}^{-1}(\mathbb{A})]\!]\right) & (\mathbf{f} \text{ is box conservative}) \\ & = & [\![\mathbf{f}^{-1}(\mathbb{A})]\!]. \ ■ & \end{array} \tag{19}$$

Example 3. A rotation from $\mathbb{R}^2 \to \mathbb{R}^2$ of angle α is box-conservative iff $\alpha = k \cdot \frac{\pi}{2}$, $k \in \mathbb{Z}$.

Definition 2. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective function, we define the image by $\mathbf{f}$ of a contractor as follows:

$$\mathbf{f}(\mathcal{C}_{\mathbb{X}}) = \mathbf{f} \circ \mathcal{C}_{\mathbb{X}} \circ \mathbf{f}^{-1}. \tag{20}$$

This new definition, will make it possible to extend the contractor algebra to more complex operations. For instance $\left(\mathbf{f}^0 \sqcup \mathbf{f}^1 \sqcup \mathbf{f}^2\right)(\mathcal{C}_{\mathbb{X}})$ defines the following contractor

$$[\mathbf{x}] \rightarrow \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \sqcup \left(\mathbf{f} \circ \mathcal{C}_{\mathbb{X}} \circ \mathbf{f}^{-1}([\mathbf{x}])\right) \sqcup \left(\mathbf{f}^2 \circ \mathcal{C}_{\mathbb{X}} \circ \mathbf{f}^{-2}([\mathbf{x}])\right). \tag{21}$$

This contractor is consistent with the set $\mathbb{X} \cup \mathbf{f}(\mathbb{X}) \cup \mathbf{f} \circ \mathbf{f}(\mathbb{X})$, as shown at least partly by the new following proposition.

Proposition 3. *Define a set* $\mathbb{X}$ *for which we have a minimal contractor* $\mathcal{C}_{\mathbb{X}}$*. If* $\mathbf{f}$ *is box-conservative, then* $\mathbf{f}(\mathcal{C}_{\mathbb{X}})$ *is the minimal contractor for* $\mathbf{f}(\mathbb{X})$*.*

PROOF. The minimal contractor for $\mathbf{f}(\mathbb{X})$ is $[\mathbf{x}] \mapsto [\![\mathbf{f}(\mathbb{X}) \cap [\mathbf{x}]]\!]$. Now,

$$\begin{array}{rcll} [\![\mathbf{f}(\mathbb{X}) \cap [\mathbf{x}]]\!] & = & \mathbf{f} \circ \mathbf{f}^{-1}([\![\mathbf{f}(\mathbb{X}) \cap [\mathbf{x}]]\!]) & (\mathbf{f} \text{ is bijective}) \\ & = & \mathbf{f}\left([\![\mathbf{f}^{-1} \circ \mathbf{f}(\mathbb{X}) \cap \mathbf{f}^{-1}([\mathbf{x}])]\!]\right) & (\mathbf{f}^{-1} \text{ is box conservative}) \\ & = & \mathbf{f}\left([\![\mathbb{X} \cap \mathbf{f}^{-1}([\mathbf{x}])]\!]\right) & (\mathbf{f}^{-1} \circ \mathbf{f}(\mathbb{X}) = \mathbb{X}) \\ & = & \mathbf{f}\left(\mathcal{C}_{\mathbb{X}}\left(\mathbf{f}^{-1}([\mathbf{x}])\right)\right) & (\text{minimality of } \mathcal{C}_{\mathbb{X}}) \\ & = & (\mathbf{f}(\mathcal{C}_{\mathbb{X}}))([\mathbf{x}]) & (\text{definition of } \mathbf{f}(\mathcal{C}_{\mathbb{X}})) \end{array} \tag{22}$$

Thus $\mathbf{f}(\mathcal{C}_{\mathbb{X}}) = \mathbf{f} \circ \mathcal{C}_{\mathbb{X}} \circ \mathbf{f}^{-1}$ is the minimal contractor for $\mathbf{f}(\mathbb{X})$. ■

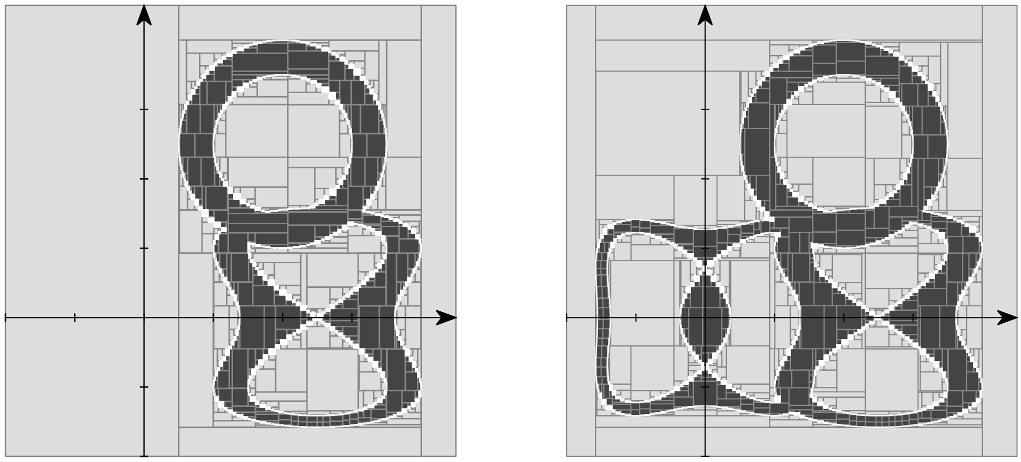
Corrolary 1. *If* $\mathbf{f}$ *is box conservative and if* $\mathbf{I}$ *is the identity function, from Proposition 3, the minimal contractor for the set* $\mathbb{X} \cup \mathbf{f}(\mathbb{X})$ *is* $(\mathbf{I} \sqcup \mathbf{f})(\mathcal{C}_{\mathbb{X}})$

PROOF. The minimal contractor consistent with $\mathbb{X} \cup \mathbf{f}(\mathbb{X})$ is

$$\begin{array}{rcll} [\![(\mathbb{X} \cup \mathbf{f}(\mathbb{X})) \cap [\mathbf{x}]]\!] & = & [\![(\mathbb{X} \cap [\mathbf{x}]) \cup (\mathbf{f}(\mathbb{X}) \cap [\mathbf{x}])]\!] & ((\mathbb{A} \cup \mathbb{B}) \cap \mathbb{C} = (\mathbb{A} \cap \mathbb{C}) \cup (\mathbb{B} \cap \mathbb{C})) \\ & = & [\![\mathbb{X} \cap [\mathbf{x}]]\!] \sqcup [\![\mathbf{f}(\mathbb{X}) \cap [\mathbf{x}]]\!] & ([\![\mathbb{A} \cup \mathbb{B}]\!] = [\![\mathbb{A}]\!] \sqcup [\![\mathbb{B}]\!]) \\ & = & \mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \sqcup \mathbf{f}(\mathcal{C}_{\mathbb{X}})([\mathbf{x}]) & \\ & = & (\mathcal{C}_{\mathbb{X}} \sqcup \mathbf{f}(\mathcal{C}_{\mathbb{X}}))([\mathbf{x}]) & \\ & = & ((\mathbf{I} \sqcup \mathbf{f})(\mathcal{C}_{\mathbb{X}}))([\mathbf{x}]). & \blacksquare \end{array}$$

Corrolary 2. *If* $\mathbf{f}$ *is box-conservative and if* $\mathcal{S}_{\mathbb{X}} = \left\{\mathcal{S}_{\mathbb{X}}^{in}, \mathcal{S}_{\mathbb{X}}^{out}\right\}$ *is the minimal separator for* $\mathbb{X}$*, then the minimal separator for* $\mathbf{f}(\mathbb{X})$ *is*

$$\mathbf{f}(\mathcal{S}_{\mathbb{X}}) = \left\{\mathbf{f} \circ \mathcal{S}_{\mathbb{X}}^{in} \circ \mathbf{f}^{-1}, \mathbf{f} \circ \mathcal{S}_{\mathbb{X}}^{out} \circ \mathbf{f}^{-1}\right\}. \tag{23}$$



(a) $\mathbb{X} \cup \mathbf{f}(\mathbb{X})$ (b) $\mathbb{X} \cup \mathbf{f}(\mathbb{X}) \cup \mathbf{f}^2(\mathbb{X})$

Figure 4: The transformation of a minimal separator by a box-conservative function is minimal.

PROOF. This is a direct consequence of Proposition 3. ■

Example 4. Let us consider the set $\mathbb{X}$ defined in Example 1, and the following box-conservative function:

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \sqrt{3 - x_2} \\ x_1 \end{pmatrix}. \tag{24}$$

From a minimal separator for the set $\mathbb{X}$ as defined in Example 2, Corollary 2 allows us to obtain minimal separators consistent with the sets $\mathbb{X} \cup \mathbf{f}(\mathbb{X})$ and $\mathbb{X} \cup \mathbf{f}(\mathbb{X}) \cup \mathbf{f}^2(\mathbb{X})$. The corresponding subpavings are depicted on Figure 4.

3. Building Minimal Contractors

Building minimal contractors for set defined by inequalities can sometimes be obtained using interval based methods [1]. In the special case where each variable occurs only once in the expression and when all involved operators are continuous, for instance with the constraints $a + \sin(b + c \cdot d) = 0$, a simple interval evaluation followed by a backward propagation in the syntactic tree of the constraint provides a minimal contraction [3]. When the constraint is monotonic with respect to all variables, then again, it is possible to reach the minimality [8]. In this section, we propose some new results that will allow us to extend the class of constraints for which we can provide a minimal contractor.

3.1. Minimal contractors

We consider here an equation of the form $\mathbf{y} = \mathbf{f}(\mathbf{x})$. The following new theorem defines a new way to build minimal contractors for this equation.

Theorem 1. *The minimal contractor consistent with* $\mathbb{S} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{y} = \mathbf{f}(\mathbf{x})\}$*, where* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^p$*, is*

$$\mathcal{C}_{\mathbb{S}} \begin{pmatrix} [\mathbf{x}] \\ [\mathbf{y}] \end{pmatrix} = \begin{pmatrix} [\![[\mathbf{x}] \cap \mathbf{f}^{-1}([\mathbf{y}])]\!] \\ [\![[\mathbf{y}] \cap \mathbf{f}([\mathbf{x}])]\!] \end{pmatrix}. \quad (25)$$

Proof. Denote $\mathbf{x}_{\neq i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $[\mathbf{x}_{\neq i}] = [x_1] \times \cdots \times [x_{i-1}] \times [x_{i+1}] \times \cdots \times [x_n]$ where the $[x_i]$'s are the interval components of the box $[\mathbf{x}]$. The optimal contraction for x_i is

$$\begin{aligned} & [\![\{x_i \in [x_i] \mid \exists \mathbf{x}_{\neq i} \in [\mathbf{x}_{\neq i}], \exists \mathbf{y} \in [\mathbf{y}], \mathbf{y} = \mathbf{f}(\mathbf{x})\}]\!] && (26) \\ = \; & [\![\{x_i \in [x_i], \exists \mathbf{x}_{\neq i} \in [\mathbf{x}_{\neq i}], \mathbf{x} \in \mathbf{f}^{-1}([\mathbf{y}])\}]\!] && (27) \\ = \; & [\![\mathrm{proj}_{x_i}([\mathbf{x}] \cap \mathbf{f}^{-1}([\mathbf{y}]))]\!]. && (28) \end{aligned}$$

Now, since for any subset of $\mathbb{R}^n$:

$$[\![\mathbb{A}]\!] = [\![\mathrm{proj}_{x_1}(\mathbb{A})]\!] \times \cdots \times [\![\mathrm{proj}_{x_n}(\mathbb{A})]\!], \quad (29)$$

we get

$$[\![[\mathbf{x}] \cap \mathbf{f}^{-1}([\mathbf{y}])]\!] = [\![\mathrm{proj}_{x_1}([\mathbf{x}] \cap \mathbf{f}^{-1}([\mathbf{y}]))]\!] \times \cdots \times [\![\mathrm{proj}_{x_n}([\mathbf{x}] \cap \mathbf{f}^{-1}([\mathbf{y}]))]\!]. \quad (30)$$

Let us apply the same reasoning with y_i. The optimal contraction for y_i is

$$\begin{aligned} & [\![\{y_i \in [y_i] \mid \exists \mathbf{y}_{\neq i} \in [\mathbf{y}_{\neq i}], \exists \mathbf{x} \in [\mathbf{x}], \mathbf{y} = \mathbf{f}(\mathbf{x})\}]\!] && (31) \\ = \; & [\![\{y_i \in [y_i] \mid \exists \mathbf{y}_{\neq i} \in [\mathbf{y}_{\neq i}], \mathbf{y} \in \mathbf{f}([\mathbf{x}])\}]\!] && (32) \\ = \; & [\![\mathrm{proj}_{y_i}([\mathbf{y}] \cap \mathbf{f}([\mathbf{x}]))]\!]. && (33) \end{aligned}$$

Thus

$$[\![[\mathbf{y}] \cap \mathbf{f}([\mathbf{x}])]\!] = [\![\text{proj}_{y_1}([\mathbf{y}] \cap \mathbf{f}([\mathbf{x}]))]\!] \times \cdots \times [\![\text{proj}_{y_m}([\mathbf{y}] \cap \mathbf{f}([\mathbf{x}]))]\!] . \tag{34}$$

As a consequence, $\mathcal{C}_{\mathbb{S}}$ corresponds to the minimal contractor consistent with $\mathbb{S}$.■

3.2. Polar contractor

The results given in the previous section are applied here to build a minimal contractor for the *polar set* defined by:

$$\Pi = \left\{ \mathbf{p} = (x, y, \rho, \theta) \in \mathbb{R}^4, (x, y) = \boldsymbol{\pi}(\rho, \theta) \right\} \tag{35}$$

where

$$\boldsymbol{\pi} \begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} \rho \cos \theta \\ \rho \sin \theta \end{pmatrix} \tag{36}$$

is the *polar function.* Define

$$\Pi_0 = [\mathbf{p}_0] \cap \Pi \tag{37}$$

with

$$[\mathbf{p}_0] = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \left[0, \frac{\pi}{4}\right]. \tag{38}$$

On $[\mathbf{p}_0]$, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{\pi} \begin{pmatrix} \rho \\ \theta \end{pmatrix} \Leftrightarrow \begin{cases} \rho & = & \sqrt{x^2 + y^2} \\ \theta & = & \text{atan}(\frac{y}{x}) \end{cases} \tag{39}$$

i.e.,

$$\begin{pmatrix} \rho \\ \theta \end{pmatrix} = \boldsymbol{\pi}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2} \\ \text{atan}(\frac{y}{x}) \end{pmatrix}. \tag{40}$$

From Theorem 1, the minimal contractor for Π_0 is

$$\mathcal{C}_{\Pi_0} \begin{pmatrix} [x] \times [y] \\ [\rho] \times [\theta] \end{pmatrix} = \begin{pmatrix} [\![[x] \times [y] \cap \boldsymbol{\pi}([\rho] \times [\theta])]\!] \\ [\![[\rho] \times [\theta] \cap \boldsymbol{\pi}^{-1}([x] \times [y])]\!] \end{pmatrix} \tag{41}$$

Figure 5 illustrates the contraction of five different boxes $[x] \times [y] \times [\rho] \times [\theta]$. The light gray pies is the initial domain for $[\rho]$ and $[\theta]$ while the dark gray pies are the resulting domains obtained after contraction. For instance, the box $[x_1] \times [y_1] \times [\rho] \times [\theta]$ on the left in contracted into the box $[x_1] \times [y_1] \times [\rho_1] \times [\theta_1]$ on the right.

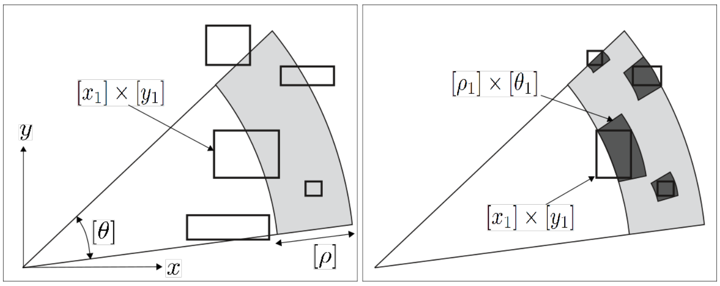


Figure 5: Left: before contraction, Right: after contraction.

The minimal contractor $\mathcal{C}_\Pi$ for Π can be deduced from $\mathcal{C}_{\Pi_0}$ using the following proposition.

Proposition 4. *Define the following symmetries*

$$\begin{aligned} \sigma_1 : (x, y, \rho, \theta) &\rightarrow \left(y, x, \rho, \frac{\pi}{2} - \theta\right) &(42)\\ \sigma_2 : (x, y, \rho, \theta) &\rightarrow (x, -y, \rho, -\theta) &(43)\\ \sigma_3 : (x, y, \rho, \theta) &\rightarrow (-x, y, \rho, \pi - \theta) &(44)\\ \sigma_4 : (x, y, \rho, \theta) &\rightarrow (x, y, -\rho, \pi + \theta) &(45)\\ \gamma : (x, y, \rho, \theta) &\rightarrow (x, y, \rho, \theta + 2\pi)\,. &(46) \end{aligned}$$

A minimal contractor for Π *is*

$$\mathcal{C}_\Pi = \left(\bigsqcup_i \gamma^i \circ (\mathbf{I} \sqcup \sigma_4) \circ (\mathbf{I} \sqcup \sigma_3) \circ (\mathbf{I} \sqcup \sigma_2) \circ (\mathbf{I} \sqcup \sigma_1) \right) \ (\mathcal{C}_{\Pi_0})\,. \tag{47}$$

Proof. By composing transformation functions, the initial domain, restricted to $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times [0, \frac{\pi}{4}]$ is extended to $\mathbb{R}^4$. We have

$$\mathcal{C}_{\Pi_0} \sim \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \left[0, \frac{\pi}{4}\right] \cap \Pi$$
$$(\mathbf{I} \sqcup \sigma_1)\, \mathcal{C}_{\Pi_0} \sim \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \left[0, \frac{\pi}{2}\right] \cap \Pi$$
$$(\mathbf{I} \sqcup \sigma_2) \circ (\mathbf{I} \sqcup \sigma_1)\, \mathcal{C}_{\Pi_0} \sim \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cap \Pi$$
$$\bigcirc_{i\in\{3,2,1\}} (\mathbf{I} \sqcup \sigma_i)\, \mathcal{C}_{\Pi_0} \sim \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times [-\pi, \pi] \cap \Pi$$
$$\bigcirc_{i\in\{4,3,2,1\}} (\mathbf{I} \sqcup \sigma_i)\, \mathcal{C}_{\Pi_0} \sim \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [-\pi, \pi] \cap \Pi$$
$$\bigsqcup_i \gamma \circ \bigcirc_{i\in\{4,3,2,1\}} (\mathbf{I} \sqcup \sigma_i)\, \mathcal{C}_{\Pi_0} \sim \Pi.$$

The minimality is a consequence of the fact that all transformations are box-conservative. ■

4. Building Minimal Separators

The previous section introduced a new way to build a minimal contractor related to constraints which are built by composition of elementary constraints and box-conservative transformations. We now extend these results to build minimal separators.

4.1. Minimal separators

Consider the set

$$\mathbb{X} = \{\mathbf{x} \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}), \tag{48}$$

where $\mathbf{f}$ is a function mapping $\mathbb{R}^n$ into $\mathbb{R}^m$. We assume here that $\mathbb{Y}$ is a subpaving (a finite union of boxes) which may overlap, *i.e.*,

$$\mathbb{Y} = \bigcup_i [\mathbf{y}](i). \tag{49}$$

Since $\mathbb{Y}$ is a subpaving, its complementary set $\overline{\mathbb{Y}}$ is also a subpaving. For instance, in Figure 4.1, we have

$$\begin{array}{rcl} \mathbb{Y} & = & [1,2] \times [1,3] \cup [3,4] \times [1,3] \\ \overline{\mathbb{Y}} & = & (\mathbb{R} \times [3,\infty]) \cup (\mathbb{R} \times [-\infty,1]) \\ & & \cup ([-\infty,1] \times [1,3]) \cup ([2,3] \times [1,3]) \\ & & \cup ([4,\infty] \times [1,3]). \end{array} \tag{50}$$

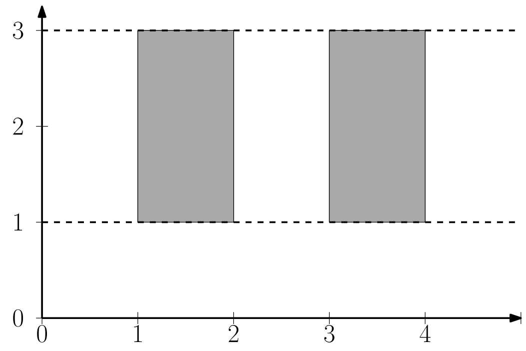


Figure 6: $\mathbb{Y}$ in gray and $\overline{\mathbb{Y}}$ can be represented by the union of boxes

Definition 3. Consider a contractor $\mathcal{C}([\mathbf{x}],[\mathbf{y}])$. We define the partial contractor with respect to $\mathbf{x}$ as the projection of the box $\mathcal{C}([\mathbf{x}],[\mathbf{y}])$ onto $\mathbf{x}$, *i.e.*,

$$\partial_{\mathbf{x}}\mathcal{C}([\mathbf{x}],[\mathbf{y}]) = [\mathbf{a}] \text{ and } \partial_{\mathbf{y}}\mathcal{C}([\mathbf{x}],[\mathbf{y}]) = [\mathbf{b}]\,, \tag{51}$$

$$\text{where } ([\mathbf{a}],[\mathbf{b}]) = \mathcal{C}([\mathbf{x}],[\mathbf{y}]). \tag{52}$$

Theorem 2. *Denote by $\mathcal{C}\left([\mathbf{x}],[\mathbf{y}]\right)$, the minimal a contractor consistent with the constraint $\mathbf{f}\left(\mathbf{x}\right)=\mathbf{y}$. If $\mathbb{Y}$ is a subpaving, then the minimal separator consistent with the set $\mathbb{X}=\mathbf{f}^{-1}\left(\mathbb{Y}\right)$, is*

$$\mathcal{S}\left([\mathbf{x}]\right)=\left\{\mathcal{S}^{in},\mathcal{S}^{out}\right\}\left([\mathbf{x}]\right)=\left\{\bigsqcup_{[\mathbf{y}]\in\overline{\mathbb{Y}}}\partial_{\mathbf{x}}\mathcal{C}\left([\mathbf{x}],[\mathbf{y}]\right),\bigsqcup_{[\mathbf{y}]\in\mathbb{Y}}\partial_{\mathbf{x}}\mathcal{C}\left([\mathbf{x}],[\mathbf{y}]\right)\right\}. \tag{53}$$

Proof. To prove that $\mathcal{S}$ is minimal, it suffices to prove that the two contractors $\mathcal{S}^{\text{in}}$ and $\mathcal{S}^{\text{out}}$ are minimal. Let us first prove that $\mathcal{S}^{\text{out}}$ is minimal. Define

$$\mathbb{X}=\left\{\mathbf{x}\mid\mathbf{f}\left(\mathbf{x}\right)\in\mathbb{Y}\right\}=\mathbf{f}^{-1}\left(\mathbb{Y}\right). \tag{54}$$

For a given box $[\mathbf{x}]$, the minimal contractor yields $[\![\mathbb{X}\cap[\mathbf{x}]]\!]$. Now

$$\begin{aligned}
[\![\mathbb{X}\cap[\mathbf{x}]]\!] &= [\![\mathbf{f}^{-1}(\mathbb{Y})\cap[\mathbf{x}]]\!] && \text{(definition of } \mathbb{X})\\
&= \left[\!\!\left[\mathbf{f}^{-1}(\textstyle\bigcup_{[\mathbf{y}]\in\mathbb{Y}}[\mathbf{y}])\cap[\mathbf{x}]\right]\!\!\right] && \text{(definition of } \mathbb{Y})\\
&= \left[\!\!\left[\left(\textstyle\bigcup_{[\mathbf{y}]\in\mathbb{Y}}\mathbf{f}^{-1}([\mathbf{y}])\right)\cap[\mathbf{x}]\right]\!\!\right] && (\mathbf{f}^{-1}(\mathbb{A}\cup\mathbb{B})=\mathbf{f}^{-1}(\mathbb{A})\cup\mathbf{f}^{-1}(\mathbb{B}))\\
&= \left[\!\!\left[\textstyle\bigcup_{[\mathbf{y}]\in\mathbb{Y}}\left(\mathbf{f}^{-1}([\mathbf{y}])\cap[\mathbf{x}]\right)\right]\!\!\right] && ((\mathbb{A}\cup\mathbb{B})\cap[\mathbf{x}]=(\mathbb{A}\cap[\mathbf{x}])\cup(\mathbb{B}\cap[\mathbf{x}]))\\
&= \left[\!\!\left[\textstyle\bigcup_{[\mathbf{y}]\in\mathbb{Y}}[\![\mathbf{f}^{-1}([\mathbf{y}])\cap[\mathbf{x}]]\!]\right]\!\!\right] && ([\![\mathbb{A}\cup\mathbb{B}]\!]=[\![[\![\mathbb{A}]\!]\cup[\![\mathbb{B}]\!]]\!])\\
&= \left[\!\!\left[\textstyle\bigcup_{[\mathbf{y}]\in\mathbb{Y}}\partial_{\mathbf{x}}\mathcal{C}([\mathbf{x}],[\mathbf{y}])\right]\!\!\right] && \text{(minimality of } \mathcal{C}_{[\mathbf{y}]})\\
&= \textstyle\bigsqcup_{[\mathbf{y}]\in\mathbb{Y}}\partial_{\mathbf{x}}\mathcal{C}([\mathbf{x}],[\mathbf{y}]) && \text{(definition of } \sqcup)\\
&= \mathcal{S}^{\text{out}}([\mathbf{x}]).
\end{aligned}$$

Let us now prove that $\mathcal{S}^{\text{in}}$ is minimal. Define

$$\overline{\mathbb{X}}=\{\mathbf{x}\mid\mathbf{f}(\mathbf{x})\notin\mathbb{Y}\}=\mathbf{f}^{-1}\left(\overline{\mathbb{Y}}\right). \tag{55}$$

A reasoning, similar to the first part of the proof, gives us

$$[\![\overline{\mathbb{X}}\cap[\mathbf{x}]]\!]=\bigsqcup_{[\mathbf{y}]\in\overline{\mathbb{Y}}}\partial_{\mathbf{x}}\mathcal{C}([\mathbf{x}],[\mathbf{y}])=\mathcal{S}^{\text{in}}([\mathbf{x}]), \tag{56}$$

which terminates the proof. ■

Remark 1. This theorem shows that getting the optimal set inversion mostly depends on the function $\mathbf{f}$ and not on $\mathbb{Y}$. As a consequence, it is worthwhile to spend some times to compute optimal contractors for the constraint $\mathbf{f}(\mathbf{x})=\mathbf{y}$ in order to derive minimal separator for $\mathbf{f}(\mathbf{x})\in\mathbb{Y}$.

4.2. Polar Separator

We consider the problem of approximating to the projection of Π (see Equation (35)) along the (ρ,θ) axis defined by:

$$[\rho]\times[\theta]\cap\boldsymbol{\pi}^{-1}([x]\times[y]). \tag{57}$$

Let $\mathcal{C}_{\Pi}$ be the minimal polar contractor consistent with Π. From Theorem 2, the minimal separator consistent with this set is

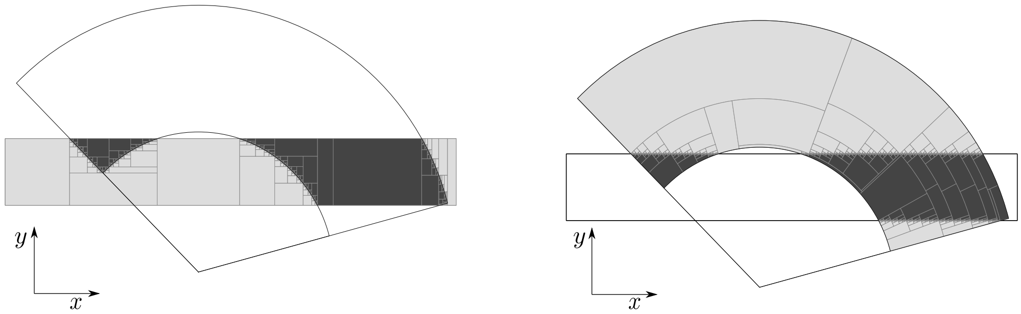


Figure 7: Action of the two separators defined as the projection of the polar contractor. The initial domains for (x, y) and (ρ, θ) are represented by the large rectangle and the large pie. Dark gray boxes and pies belong to solutions sets while those in light gray do not

$$\mathcal{S}_{\pi}^{[x],[y]}\left([\rho],[\theta]\right)=\left\{\bigsqcup_{[x_1]\times[y_1]\in\overline{[x]\times[y]}}\partial_{\rho\theta}\mathcal{C}_{\Pi}\left([x_1],[y_1],[\rho],[\theta]\right),\ \partial_{\rho\theta}\mathcal{C}_{\Pi}\left([x],[y],[\rho],[\theta]\right)\right\}. \tag{58}$$

where $\overline{[x]\times[y]}$ is a subpaving corresponding to the complementary of the box $[x]\times[y]$. Here, $[x],[y]$ are the parameters of the separators and the contractions operate on the box $[\rho]\times[\theta]$. The same reasoning applied on ρ and θ, concludes that minimal separator consistent with the set:

$$[x]\times[y]\cap\boldsymbol{\pi}\left([\rho]\times[\theta]\right) \tag{59}$$

is given by

$$\mathcal{S}_{\pi^{-1}}^{[\rho],[\theta]}\left([x],[y]\right)=\left\{\bigsqcup_{[\rho_1]\times[\theta_1]\in\overline{[\rho]\times[\theta]}}\partial_{xy}\mathcal{C}_{\Pi}\left([x],[y],[\rho_1],[\theta_1]\right),\ \partial_{xy}\mathcal{C}_{\Pi}\left([x],[y],[\rho],[\theta]\right)\right\}. \tag{60}$$

where $\overline{[\rho]\times[\theta]}$ is a subpaving corresponding to the complementary of the pie $[\rho]\times[\theta]$. The efficiency of these two separators are illustrated on Figure 7 with the initial intervals taken as $[x]=[-1,4]$,$[y]=[1,2]$, $[\rho]=[2.1,4]$ and $[\theta]=[\frac{\pi}{12},\frac{3\pi}{4}]$. The minimality of the separators can be observed by the fact that each box intersects the border of the pie in Figure 7 (left) and each pie intersects the boundary of the box in Figure 7 (right).

5. Application to Localization

Sensors which return range and goniometric measurements of a given landmark, such as sonar or LIDAR, are commonly used in robotics for localization. When measurements are related to some landmarks, the problem can be modeled using the polar constraint [16]. We propose now examples of localization to illustrate the efficiency of the polar separator.

5.1. Localization with one landmark

Consider one landmark at a known position $\mathbf{m} = (m_1, m_2)$. One robot at position $\mathbf{x} = (x_1, x_2)$ is able to measure the distance y_1 to $\mathbf{m}$ and the direction y_2 of $\mathbf{m}$ in the local reference frame with an interval accuracy. In this example, we take $[\mathbf{y}] = [4, 6] \times \left[-\frac{\pi}{12}, \frac{\pi}{4}\right]$. The set $\mathbb{X}$ of feasible $\mathbf{x}$ is:

$$\begin{aligned} \mathbb{X} &= \left\{\mathbf{x} \in \mathbb{R}^2 \mid \exists \mathbf{y} \in [\mathbf{y}], \mathbf{m} - \mathbf{x} = \boldsymbol{\pi}(\mathbf{y})\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{m} - \mathbf{x} \in \boldsymbol{\pi}([\mathbf{y}])\right\} \\ &= \mathbf{g}_{\mathbf{m}} \circ \boldsymbol{\pi}\left([\mathbf{y}]\right) \end{aligned} \tag{61}$$

where $\boldsymbol{\pi}$ is defined in Equation (36) and

$$\mathbf{g}_{\mathbf{m}}\left(\mathbf{x}\right) = \mathbf{m} - \mathbf{x} \tag{62}$$

which is box-conservative. From Corollary 2, the minimal separator $\mathcal{S}_{\mathbf{m}}$, associated with landmark $\mathbf{m}$, consistent with $\mathbb{X}$ is

$$\mathcal{S}_{\mathbf{m}} = \mathbf{g}_{\mathbf{m}} \circ \mathcal{S}^{[\mathbf{y}]}_{\pi^{-1}}, \tag{63}$$

where $\mathcal{S}^{[\mathbf{y}]}_{\pi^{-1}}$ is the minimal separator defined in Equation (60).

Figure 8 shows the results of the set inversion using our minimal separator (left) and a forward-backward separator (right). Note that the new Theorem 2 has to be used in order to be able get an inner approximation, even with the forward/backward contractor.

5.2. Example with several landmarks

Assume now that we have 3 landmarks $\mathbf{m}_i$. The measurements $[\mathbf{y}_i]$ and the position of the landmarks are given in Table 5.2. The feasible set $\mathbb{X}$ is now

$$\mathbb{X} = \bigcap_i \mathbf{g}_{\mathbf{m}_i} \circ \boldsymbol{\pi}\left([\mathbf{y}_i]\right). \tag{64}$$

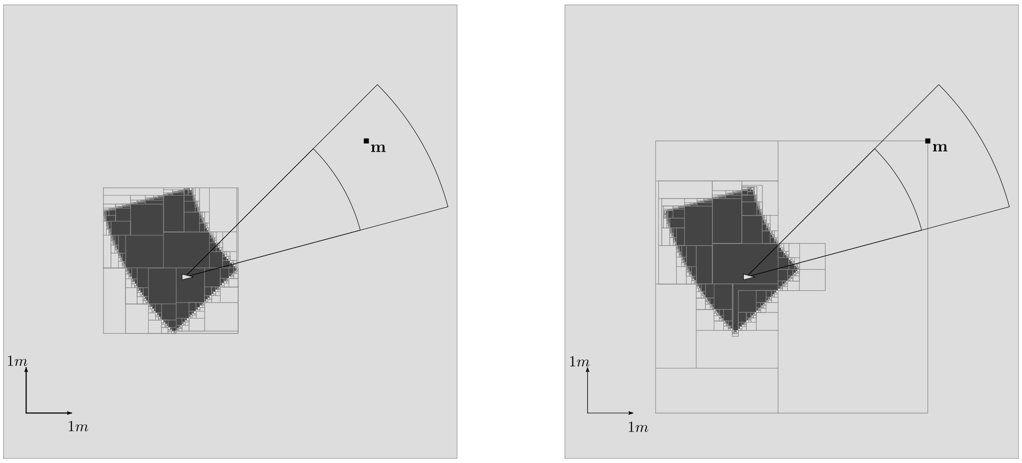


Figure 8: Approximation of $\mathbb{X}$ using the minimal separator (Left) and by using a classical forward-backward separator (Right). The initial domain is $\mathbb{X}_0 = [-6, 4]^2$.

landmarks	1	2	3
m_1	6	-2	-3
m_2	12	-5	10
$[y_1]$(m)	[11, 12]	[8, 10]	[5, 7]
$[y_2]$(°)	[14, 100]	[-147, -75]	[63, 150]

Table 2: Range and bearing measurements of the three landmarks

The corresponding separator is

$$\mathcal{S}_{\mathbb{X}} = \bigcap_i \mathbf{g}_{\mathbf{m}_i} \circ \mathcal{S}_{\pi^{-1}}^{[\mathbf{y}_i]}. \tag{65}$$

The paving obtained using $\mathcal{S}_{\mathbb{X}}$ is shown in Figure 5.2. The black dashed boxes corresponds to the result of the first call of the forward/backward separator and is bigger than the one obtained with $\mathcal{S}$.

Remark 2. Contrary to the union of separators, the intersection of minimal separators is not minimal, so $\mathcal{S}$ is not minimal. Now, the inner contractor of $\mathcal{S}$, which is an union of minimal contractors, is still minimal. This can be observed on Figure 5.2.

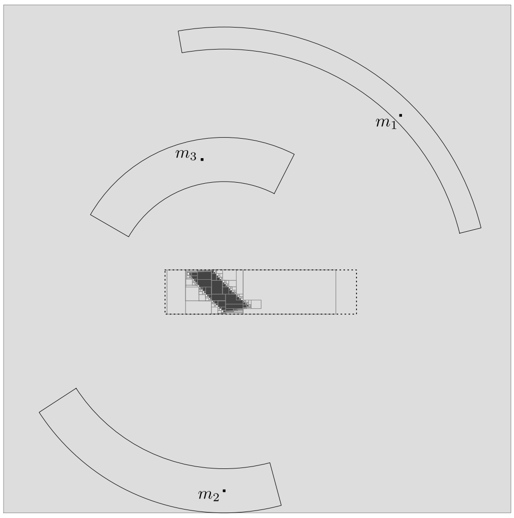


Figure 9: Paving obtained using the polar separator. The dashed black box is the result of the first call a a forward/backward separator

5.3. Underwater Localization

We now illustrate the efficiency of the polar separator on a data-set extracted from a mission performed by the actual AUV (Autonomous Underwater Vehicle) VAMA (see Figure 10) in the Road-stead of Brest (France, Brittany), July 18, 2015. We focus on the transit phase of the mission where the AUV follows a set of way points in order to reach it mission area. During this phase, ranges and bearings $\mathbf{y}_k = (y_1^k, y_2^k)$ between the ship at position $\mathbf{m} = (m_1, m_2)$ and the AUV at time k are periodically measured using an *ultra short baseline* (USBL) and then sent to the robot through an acoustic communication. We assume that no communication delays exists. Moreover, thanks to the pressure sensor, the robots knows its depth and the localization problem can be easily be projected on the two-dimensional horizontal plane.

An estimated reconstitution of the part of the mission made by VAMA that will be used here is depicted on Figure 11. The motion of the robot is assumed to be described by the discrete-time state equation:

$$\mathbf{x}(k+1) = \boldsymbol{\varphi}_k(\mathbf{x}(k)) = \mathbf{x}(k) + \begin{pmatrix} \cos\psi(k) & -\sin\psi(k) \\ \sin\psi(k) & \cos\psi(k) \end{pmatrix} \cdot \mathbf{v}(k), \qquad (66)$$

where $\mathbf{x}(k)$ corresponds to the 2D coordinates of the center of the robot at time k

Figure 10: VAMA (Véhicule Anti Mine Autonome), an AUV owned by DGA Tn.

expressed in an absolute inertial frame, $\psi(k)$ is the heading and $\mathbf{v}(k)$ is the horizontal speed vector of the robot expressed in its own coordinate system. The speed and heading are measured using a MEMS IMU (Xsens Mti) and are assumed to be known with an uncertainty of $\pm 0.05\, m.s^{-1}$ and $\pm 1°$. The range is measured with a accuracy of $\pm 1\%$ and the bearing with an accuracy of $\pm 2°$.

As a state estimator, we propose to compute the set $\mathbb{X}_k$ of feasible state $\mathbf{x}$ consistent with the last $\bar{i}$ measurements. It is given by

$$\mathbb{X}_k = \left\{\mathbf{x} \in \mathbb{R}^2 \mid \forall i \in \{0, \bar{i}\}, \exists \mathbf{y} \in [\mathbf{y}_i], \mathbf{m}_{k-i} - \boldsymbol{\varphi}_{k:k-i}(\mathbf{x}) = \boldsymbol{\pi}(\mathbf{y}_{k-i})\right\} \tag{67}$$

where $\boldsymbol{\varphi}_{k_1:k_2}$ is the flow defined as follows

$$\begin{cases} \boldsymbol{\varphi}_{k_1:k_2} = \boldsymbol{\varphi}_{k_2-1:k_2} \circ \cdots \circ \boldsymbol{\varphi}_{k_1:k_1+1} & if\, k_2 \geq k_1 \\ \boldsymbol{\varphi}_{k_1:k_2} = \boldsymbol{\varphi}^{-1}_{k_2:k_1} & \text{otherwise.} \end{cases} \tag{68}$$

We have

$$\begin{aligned} \mathbb{X}_k &= \bigcap_{i \in \{0,\bar{i}\}} \left\{\mathbf{x} \in \mathbb{R}^2 \mid \exists \mathbf{y} \in [\mathbf{y}_{k-i}], \mathbf{m}_{k-i} - \boldsymbol{\varphi}_{k:k-i}(\mathbf{x}) = \boldsymbol{\pi}(\mathbf{y}_{k-i})\right\} \\ &= \bigcap_{i \in \{0,\bar{i}\}} \left\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{m}_{k-i} - \boldsymbol{\varphi}_{k:k-i}(\mathbf{x}) \in \boldsymbol{\pi}([\mathbf{y}_{k-i}])\right\} \\ &= \bigcap_{i \in \{0,\bar{i}\}} \boldsymbol{\varphi}_{k-i:k} \circ \mathbf{g}_{\mathbf{m}_{k-i}} \circ \boldsymbol{\pi}([\mathbf{y}_{k-i}]). \end{aligned} \tag{69}$$

The corresponding separator is

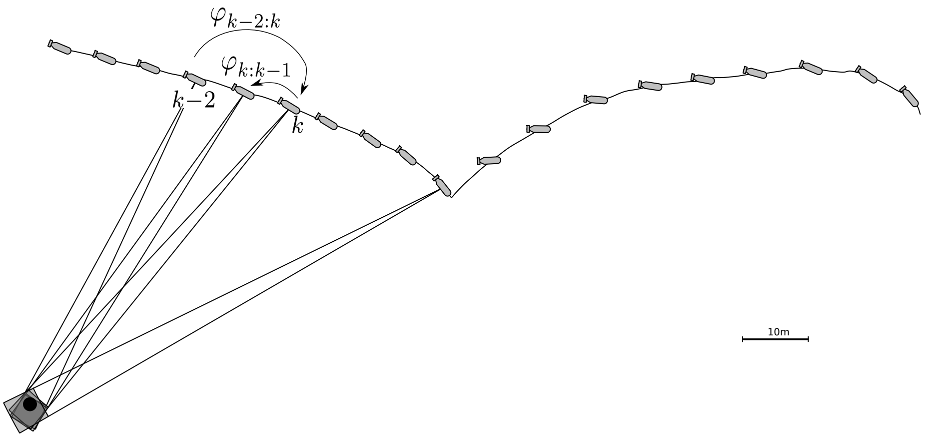


Figure 11: Part of the trajectory of VAMA. The robot changes its direction when it reached one way-point.

$$\mathcal{S}_{\mathbb{X}_k} = \bigcap_{i \in \{0,\bar{i}\}} \boldsymbol{\varphi}_{k-i:k} \circ \mathbf{g}_{\mathbf{m}_{k-i}} \circ \mathcal{S}_{\pi^{-1}}^{[\mathbf{y}_{k-i}]}. \tag{70}$$

For an implementation of the observer (70), $\mathcal{S}_{\pi^{-1}}^{[\mathbf{y}_{k-i}]}$ can either be the minimal polar separator or a separator based on a classical forward/backward propagation. For a comparison, let us first apply the separator $\mathcal{S}_{\mathbb{X}_k}$ for all k without bisection and with $\bar{i} = 0$. As expected, no inner contraction occur. We only get boxes $[\mathbf{x}_k]$ enclosing $\mathbb{X}_k$ the diameters of which are depicted in Figure 12 (blue). The figure also shows the diameter of the boxes $[\mathbf{x}_k]$ that are obtained with a forward/backward separator (green). We conclude that the forward/backward separator is indeed more pessimistic which is consistent with the fact that the polar contractor is minimal (see also the video in [11]).

Figure 13 shows an approximation approximation of $\mathbb{X}_k$ for $\bar{i} = 5$ and $t = 55\,\text{sec}$. Since there is an acoustic measurement every $5\,\text{sec}$ and since the sampling time is $\delta = 0.1\,\text{sec}$, a synchronization of all the data (not discussed here) had to be done. The correspondence between k and t is $k = 5 \cdot t$.

To be robust with respect to outliers, we may allow the observer to relax on the time window of length $\bar{i}$ at most q outliers [12]. The solution set $\mathbb{X}_k$ becomes:

$$\mathbb{X}_k = \bigcap_{i \in \{0,\bar{i}\}}^{\{q\}} \boldsymbol{\varphi}_{k-i:k} \circ \mathbf{g}_{\mathbf{m}_{k-i}} \circ \boldsymbol{\pi}\left([\mathbf{y}_{k-i}]\right) \tag{71}$$

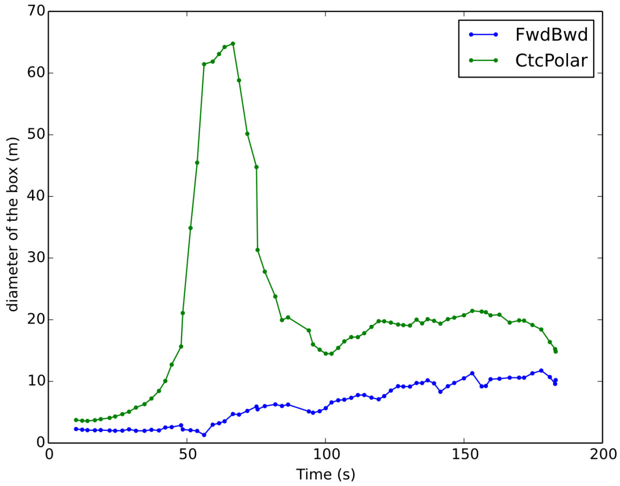


Figure 12: Diameter of the boxes $[\mathbf{x}_k]$ obtained by the minimal polar separator (blue) and the forward/backward separator (green).

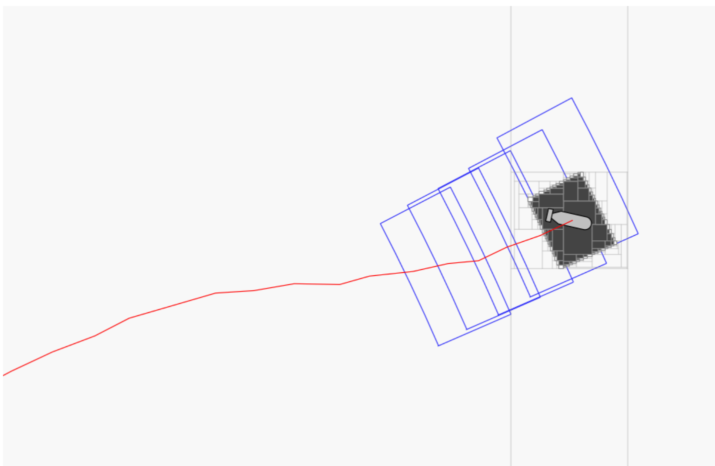


Figure 13: Localization of the robot for $t = 55\,\mathrm{sec}$. Each of the blue pie corresponds to all positions of the robot associated with one measurement.

where the q-relaxed intersection [20][5] is used. The associated separator becomes

$$\mathcal{S}_{\mathbb{X}_k} = \bigcap_{i \in \{0, \bar{i}\}}^{\{q\}} \boldsymbol{\varphi}_{k-i:k} \circ \mathbf{g}_{\mathbf{m}_{k-i}} \circ \mathcal{S}_{\pi^{-1}}^{[\mathbf{y}_{k-i}]}. \quad (72)$$

The result obtained using this observer for $q \in \{0, 1, 2\}$ are illustrated on a video given in [11].

6. Conclusion

Contractor-based techniques are particularly attractive when solving engineering applications, due to the fact that they can handle and propagate uncertainties in a context where the equations of the problem are non-linear and non-convex. Now, the performances of paving methods are extremely sensitive to the accuracy of the contractors that are consistent with the equations. One of equation which is often met in practice is the polar equation which links Cartesian to polar coordinates.

In this paper, we proposed two new theorems that could help to build more easily minimal contractors and separators consistent with some specific constraints. These theorems allowed us to build the minimal contractor and a minimal separator for the important polar constraint, which was not done before, to our knowledge.

The efficiency of these new operators and their ability to get an inner and outer approximation of the solution set was illustrated on the problem of the localization of a robot when both goniometric and range measurements are collected.

Note. The `Python` programs associated with all examples, the navigation data made by the robot VAMA and some videos illustrating the use of the polar contractor are given in [11].

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