Examen sur la commande non-linéaire des robots mobiles
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Exercise 1. Consider the robot described by

$$\begin{cases} \dot{x}_1 &= x_4 \cos x_3 \\ \dot{x}_2 &= x_4 \sin x_3 \\ \dot{x}_3 &= u_1 \\ \dot{x}_4 &= u_2 \end{cases}$$

where (x_1, x_2) corresponds to the position of the cart, x_3 to its heading and x_4 to its speed.

1) Provide a controller based on a feedback linearization to make the robot follows the trajectory:

$$\begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix} = \begin{pmatrix} t \\ \sin 3t \end{pmatrix}.$$

2) Provide a sliding mode controller which makes the robot following the same trajectory.

Exercise 2. Consider a group of m robots the motion of which is described by the state equation

$$\begin{cases} \dot{x}_1 &= x_4 \cos x_3 \\ \dot{x}_2 &= x_4 \sin x_3 \\ \dot{x}_3 &= u_1 \\ \dot{x}_4 &= u_2 \end{cases}$$

where (x_1, x_2) corresponds to the position of the cart, x_3 to its heading and x_4 to its speed.

1) Provide a controller for each of these robots so that the ith robot follows the trajectory

$$\begin{pmatrix} \cos(at + \frac{2i\pi}{m}) \\ \sin(at + \frac{2i\pi}{m}) \end{pmatrix}$$

where a is a constant. As a consequence, after the initialization step, all robots are uniformly distributed on the unit circle, turning around the origin.

2) By using a linear transformation of the unit circle, change the controllers for the robots so that all robots stay on a moving ellipse with the first axis of length $20 + 15 \cdot \sin(at)$ and the second axis of length 20. Moreover, we make the ellipse rotating by choosing an angle for the first axis of $\theta = at$.

Correction

Correction of Exercise 1

1) If we set $\mathbf{y} = (x_1, x_2)$, we have

$$\begin{cases} \ddot{y}_1 &= -\dot{x}_3 x_4 \sin x_3 + \dot{x}_4 \cos x_3 = -x_4 \sin x_3 u_1 + \cos x_3 u_2 \\ \ddot{y}_2 &= \dot{x}_3 x_4 \cos x_3 + \dot{x}_4 \sin x_3 = x_4 \cos x_3 u_1 + \sin x_3 u_2 \end{cases}$$

i.e.,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -x_4 \sin x_3 & \cos x_3 \\ x_4 \cos x_3 & \sin x_3 \end{pmatrix} \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix}.$$

To have the error dynamic

$$\mathbf{y}_d(t) - \mathbf{y}(t) + 2(\dot{\mathbf{y}}_d(t) - \dot{\mathbf{y}}(t)) + (\ddot{\mathbf{y}}_d(t) - \ddot{\mathbf{y}}(t)) = \mathbf{0},$$

we should take

$$\ddot{\mathbf{y}}(t) = \mathbf{y}_d(t) - \mathbf{y}(t) + 2(\dot{\mathbf{y}}_d(t) - \dot{\mathbf{y}}(t)) + \ddot{\mathbf{y}}_d(t).$$

The corresponding controller is thus

$$\mathbf{u} = \begin{pmatrix} -x_4 \sin x_3 & \cos x_3 \\ x_4 \cos x_3 & \sin x_3 \end{pmatrix}^{-1} \cdot (\mathbf{y}_d(t) - \mathbf{y}(t) + 2(\dot{\mathbf{y}}_d(t) - \dot{\mathbf{y}}(t)) + \ddot{\mathbf{y}}_d(t))$$

where

$$\mathbf{y}_d(t) = \begin{pmatrix} t \\ \sin 3t \end{pmatrix}, \dot{\mathbf{y}}_d(t) = \begin{pmatrix} 1 \\ 3\cos 3t \end{pmatrix}, \ddot{\mathbf{y}}_d(t) = \begin{pmatrix} 0 \\ -9\sin 3t \end{pmatrix}$$

2) Take as the desired surface:

$$\mathbf{s}(\mathbf{x}, t) = \mathbf{y}_d(t) - \mathbf{y}(t) + \dot{\mathbf{y}}_d(t) - \dot{\mathbf{y}}(t) = \mathbf{0},$$

which also corresponds to the dynamic of the error we want. Therefore

$$\mathbf{s}(\mathbf{x}, t) = \begin{pmatrix} t \\ \sin(3t) \end{pmatrix} - \mathbf{y}(t) + \begin{pmatrix} 1 \\ 3\cos(3t) \end{pmatrix} - \begin{pmatrix} x_4 \cos x_3 \\ x_4 \sin x_3 \end{pmatrix}.$$

The controller is

$$\mathbf{u} = \begin{pmatrix} -x_4 \sin x_3 & \cos x_3 \\ x_4 \cos x_3 & \sin x_3 \end{pmatrix}^{-1} (K \cdot \text{sign}(\mathbf{s}(\mathbf{x}, t)))$$

where K is large. Here, since $\mathbf{s}(\mathbf{x}, t)$ is a vector of $\mathbb{R}^2$, the function *sign* has to be understood componentwize.

Correction of Exercise 2

1) We take as an output $\mathbf{y} = (x_1, x_2)$, and we apply a feedback linearization method. We get the controller

$$\mathbf{u} = \begin{pmatrix} -x_4 \sin x_3 & \cos x_3 \\ x_4 \cos x_3 & \sin x_3 \end{pmatrix}^{-1} \cdot \left(\mathbf{c}(t) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2\dot{\mathbf{c}}(t) - 2 \begin{pmatrix} x_4 \cdot \cos(x_3) \\ x_4 \cdot \sin(x_3) \end{pmatrix} + \ddot{\mathbf{c}}(t) \right),$$

where $\mathbf{c}(t)$ is the desired position. For the ith robot, we take

$$\mathbf{c}(t) = \begin{pmatrix} \cos(at + \frac{2i\pi}{m}) \\ \sin(at + \frac{2i\pi}{m}) \end{pmatrix}, \dot{\mathbf{c}}(t) = a \cdot \begin{pmatrix} -\sin(at + \frac{2i\pi}{m}) \\ \cos(at + \frac{2i\pi}{m}) \end{pmatrix}, \ddot{\mathbf{c}}(t) = -a^2 \mathbf{c}(t) \cdot$$

2) To get the right ellipse, for each $\mathbf{c}(t)$, we apply the transformation

$$\mathbf{w}(t) = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{=\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 20 + 15 \cdot \sin(at) & 0 \\ 0 & 20 \end{pmatrix}}_{=\mathbf{D}} \cdot \mathbf{c}(t)$$

where $\mathbf{w}(t)$ is the new desired position. To apply our controller, we also need the two first derivatives of $\mathbf{w}(t)$. We have

$$\dot{\mathbf{w}} = \mathbf{R} \cdot \mathbf{D} \cdot \dot{\mathbf{c}} + \mathbf{R} \cdot \dot{\mathbf{D}} \cdot \mathbf{c} + \dot{\mathbf{R}} \cdot \mathbf{D} \cdot \mathbf{c},$$

where

$$\dot{\mathbf{D}} = \begin{pmatrix} 15a \cdot \cos(at) & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } \dot{\mathbf{R}} = a \cdot \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}.$$

Moreover

$$\ddot{\mathbf{w}} = \mathbf{R} \cdot \mathbf{D} \cdot \ddot{\mathbf{c}} + \mathbf{R} \cdot \ddot{\mathbf{D}} \cdot \mathbf{c} + \ddot{\mathbf{R}} \cdot \mathbf{D} \cdot \mathbf{c} + 2 \cdot \dot{\mathbf{R}} \cdot \mathbf{D} \cdot \dot{\mathbf{c}} + 2 \cdot \mathbf{R} \cdot \dot{\mathbf{D}} \cdot \dot{\mathbf{c}} + 2 \cdot \dot{\mathbf{R}} \cdot \dot{\mathbf{D}} \cdot \mathbf{c}$$

where

$$\ddot{\mathbf{D}} = \begin{pmatrix} -15a^2 \cdot \sin(at) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \ddot{\mathbf{R}} = -a^2 \cdot \mathbf{R}.$$

We can now apply the controller obtained at Question 1, where $\mathbf{c}(t)$ is now replaced by $\mathbf{w}(t)$.
