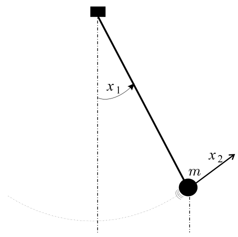
Examen sur la commande non-linéaire des robots mobiles
ENSTA-Bretagne, UV 4.7.

Jeudi 16 mai 2019, 9:30→11:00, durée 1:30,
La calculatrice est interdite. Le polycopié et les notes manuscrites sont autorisés.

Exercise 1. Consider the pendulum, represented below, and described by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1 + u \end{pmatrix}$$

where u is the input, x_1 its position and x_2 its angular velocity.



We would like the position $x_1(t)$ of the pendulum converges to the setpoint $w(t) = 0, \forall t$.

1) Taking $y = x_1$ as an output, propose a feedback linearization method to control the pendulum so that the error $e = w - y$ converges toward 0 as $\exp(-t)$.

2) Answer to the previous question with a sliding mode approach.

Exercise 2. Consider a robot moving on a plane and described by the following state equations:

$$\begin{cases} \dot{x}_1 = \cos x_3 \\ \dot{x}_2 = \sin x_3 \\ \dot{x}_3 = u \end{cases}$$

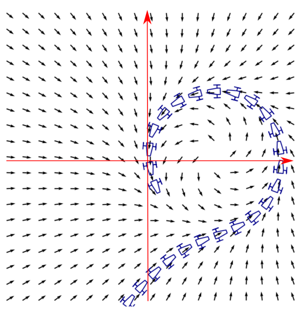
where x_3 is the heading of the robot and (x_1, x_2) are the coordinates of its center.

1) The expression of a vector field converging counterclockwise to a circle of radius 1 is given by

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \mathbf{\Phi}_0(\mathbf{p}) = \begin{pmatrix} -p_1^3 - p_1 p_2^2 + p_1 - p_2 \\ -p_2^3 - p_1^2 p_2 + p_1 + p_2 \end{pmatrix}$$

Find the expression of a vector field $\mathbf{\Phi}$ attracted by a circle of radius $\rho = 2$ and center $\mathbf{c} = (2, 0)$, where the attraction is counterclockwise.

2) Propose a controller so that the robot follows $\mathbf{\Phi}$. The wanted behavior of the controller is illustrated below.



Correction de l'examen sur la commande non-linéaire des robots mobiles
ENSTA-Bretagne, UV 4.7.

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Correction of Exercise 1.

1) We have:

$$\begin{array}{lll} y & = & x_1 \\ \dot{y} & = & x_2 \\ \ddot{y} & = & -\sin x_1 + u. \end{array}$$

If we take

$$u = \sin x_1 + (w - y + 2(\dot{w} - \dot{y}) + \ddot{w}),$$

we obtain

$$e + 2\dot{e} + \ddot{e} = 0$$

where $e = w - y$ is the error. Since $w(t) = 0, \forall t$, the controlled system is described by the following state equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - 2x_2 \end{pmatrix}.$$

2) If we are able to stay on the surface

$$s(\mathbf{x}, t) = (\dot{w} - \dot{y}) + (w - y) = 0,$$

then the error $e = w - y$ converges to zero.

To converge to the surface, we take

$$\ddot{y} = K \cdot \text{sign}(s(\mathbf{x}, t)),$$

The sliding mode controller is thus

$$\begin{array}{lll} u & = & \sin x_1 + K \cdot \text{sign}((\dot{w} - \dot{y}) + (w - y)) \\ & = & \sin x_1 - K \cdot \text{sign}(\dot{y} + y) \\ & = & \sin x_1 - K \cdot \text{sign}(x_1 + x_2) \end{array}.$$

where K is large, e.g., $K = 100$.

Correction of Exercise 2.

1) We consider have

$$\mathbf{\Phi} = \rho \cdot \mathbf{\Phi}_0 \left(\rho^{-1} \cdot (\mathbf{p} - \mathbf{c})\right)$$

2) To follow the field $\mathbf{\Phi}$, when the robot is at position $\mathbf{p} = (x_1, x_2)$, we choose for the error

$$y = x_3 - \text{atan2}(\underbrace{\Phi_2(\mathbf{p})}_{b}, \underbrace{\Phi_1(\mathbf{p})}_{a}).$$

We have

$$\begin{array}{lll} \dot{y} & = & \dot{x}_3 - (\underbrace{-\frac{b}{a^2 + b^2}}_{\frac{\partial \text{atan2}(b,a)}{\partial a}} \cdot \dot{a} + \underbrace{\frac{a}{a^2 + b^2}}_{\frac{\partial \text{atan2}(b,a)}{\partial b}} \cdot \dot{b}) \\ & = & u + \frac{b \cdot \dot{a} - a \cdot \dot{b}}{a^2 + b^2} \end{array}$$

where

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{d}{dt}\mathbf{\Phi}(\mathbf{p})$$
$$= \frac{d}{dt}\left(\rho \cdot \mathbf{\Phi}_0\left(\rho^{-1} \cdot (\mathbf{p}-\mathbf{c})\right)\right)$$
$$= \left(\frac{d\mathbf{\Phi}_0}{d\mathbf{p}}\left(\rho^{-1} \cdot (\mathbf{p}-\mathbf{c})\right)\right) \cdot \frac{d}{dt}\left((\mathbf{p}-\mathbf{c})\right)$$
$$= \left(\frac{d\mathbf{\Phi}_0}{d\mathbf{p}}\left(\frac{1}{2} \cdot \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \mathbf{c}\right)\right)\right) \cdot \begin{pmatrix} \cos x_3 \\ \sin x_3 \end{pmatrix}$$

and

$$\frac{d\mathbf{\Phi}_0}{d\mathbf{p}} = \frac{d}{d\mathbf{p}}\begin{pmatrix} -p_1^3 - p_1 p_2^2 + p_1 - p_2 \\ -p_2^3 - p_1^2 p_2 + p_1 + p_2 \end{pmatrix} = \begin{pmatrix} -3p_1^2 - p_2^2 + 1 & -2p_1 p_2 - 1 \\ -2p_1 p_2 + 1 & -3p_2^2 - p_1^2 + 1 \end{pmatrix}.$$

If we want for the dynamic of the error, $\dot{y} = -y$,we take

$$u = -y - \frac{\left(b \cdot \dot{a} - a \cdot \dot{b}\right)}{a^2 + b^2} .$$

The controller is thus

$$u \quad (\mathbf{x}) = -\text{sawtooth}\left(x_3 - \text{atan2}(b, a)\right) - \frac{\left(b \cdot \dot{a} - a \cdot \dot{b}\right)}{a^2 + b^2}$$

with

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} -3\left(\frac{x_1-2}{2}\right)^2 - \left(z_2\right)^2 + 1 & -\frac{x_2(x_1-2)}{2} - 1 \\ -\frac{x_2(x_1-2)}{2} + 1 & -3\left(\frac{x_2}{2}\right)^2 - \left(\frac{x_1-2}{2}\right)^2 + 1 \end{pmatrix} \cdot \begin{pmatrix} \cos x_3 \\ \sin x_3 \end{pmatrix}$$