Examen Inertiel, 2A Rob.

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Polycopié et notes manuscrites interdits

Durée: 60 minutes.

Consider the rotation $\mathcal{R}_{\mathbf{n},\varphi}$ of angle φ around the unit vector $\mathbf{n}$. Let $\mathbf{u}$ be a vector that we will subject to this rotation. The vector $\mathbf{u}$ can be decomposed as follows:

$$\mathbf{u} = \underbrace{<\mathbf{u},\mathbf{n}> \cdot \mathbf{n}}_{\mathbf{u}_{||}} + \underbrace{\mathbf{u} - <\mathbf{u},\mathbf{n}> \cdot \mathbf{n}}_{\mathbf{u}_{\perp}}$$

where $\mathbf{u}_{||}$ is collinear to $\mathbf{n}$ and $\mathbf{u}_{\perp}$ is in the plane $P_{\perp}$ orthogonal to $\mathbf{n}$ (see Figure 1).

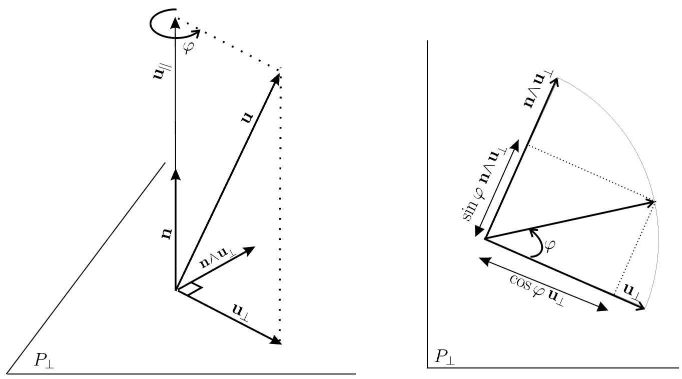

Figure 1: Rotation of the vector $\mathbf{u}$ around the vector $\mathbf{n}$; left : perspective view ; right : view from above

1) Prove Rodrigues' formula :

$$\mathcal{R}_{\mathbf{n},\varphi}(\mathbf{u}) = <\mathbf{u},\mathbf{n}> \cdot \mathbf{n} + (\cos\varphi)(\mathbf{u} - <\mathbf{u},\mathbf{n}> \cdot \mathbf{n}) + (\sin\varphi)(\mathbf{n}\wedge\mathbf{u}).$$

2) Using the double cross product formula $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \left(\mathbf{a}^{\mathrm{T}}\mathbf{c}\right)\cdot\mathbf{b} - \left(\mathbf{a}^{\mathrm{T}}\mathbf{b}\right)\cdot\mathbf{c}$, on the element $\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{u})$, show that Rodrigues' formula can also been written as:

$$\mathcal{R}_{\mathbf{n},\varphi}(\mathbf{u}) = \mathbf{u} + (1-\cos\varphi)(\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{u})) + (\sin\varphi)(\mathbf{n}\wedge\mathbf{u}).$$

Deduce from this that the matrix associated with the linear operator $\mathcal{R}_{\mathbf{n},\varphi}$ is written as:

$$\mathbf{R}_{\mathbf{n},\varphi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos\varphi)\begin{pmatrix} -n_y^2-n_z^2 & n_x n_y & n_x n_z \\ n_x n_y & -n_x^2-n_z^2 & n_y n_z \\ n_x n_z & n_y n_z & -n_x^2-n_y^2 \end{pmatrix} + (\sin\varphi)\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$

3) Conversely, we are given a rotation matrix $\mathbf{R}_{\mathbf{n},\varphi}$ for which we wish to find the axis of rotation $\mathbf{n}$ and the angle of rotation φ. Compute the trace $\mathrm{tr}(\mathbf{R}_{\mathbf{n},\varphi})$ and $\mathbf{R}_{\mathbf{n},\varphi} - \mathbf{R}_{\mathbf{n},\varphi}^{\mathrm{T}}$ and use it to obtain $\mathbf{n}$ and φ in function of $\mathbf{R}_{\mathbf{n},\varphi}$.

4) Deduce a method to find a interpolation trajectory matrix $\mathbf{R}(t)$ between two rotation matrices $\mathbf{R}_a$, $\mathbf{R}_b$ such that $\mathbf{R}(0) = \mathbf{R}_a$ and $\mathbf{R}(1) = \mathbf{R}_b$.

5) Using a Maclaurin series development of sinφ and cosφ, show that $\mathbf{R}_{\mathbf{n},\varphi} = \exp(\varphi\cdot\mathbf{n}\wedge)$