Examen Kalman, ENSI 2.

Luc Jaulin, ENSTA-Bretagne

Le vendredi 10 janvier 2020. Appareils électroniques interdits.
Polycopié et notes manuscrites autorisés
Durée: 60 minutes.

Exercise 1. Consider two Gaussian noises $\mathbf{a}$ and $\mathbf{b}$ with expectation $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ and covariances $\mathbf{\Gamma_a}, \mathbf{\Gamma_b}$. We assume that $\mathbf{a}, \mathbf{b}$ are independent.

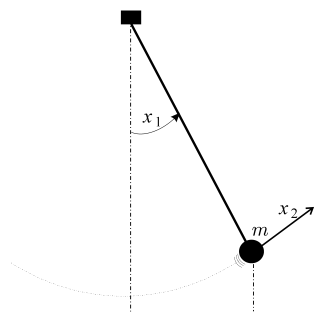
1) Write the covariance matrix $\mathbf{\Gamma_v}$ for the vector $\mathbf{v} = (\mathbf{a}, \mathbf{b})$.

2) Give the covariance matrix $\mathbf{\Gamma_c}$ of $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and for the mean $\bar{\mathbf{c}}$ with respect to $\mathbf{\Gamma_a}, \mathbf{\Gamma_b}, \bar{\mathbf{a}}, \bar{\mathbf{b}}$.

3) Give an expression for the probability density function of $\mathbf{c}$ with respect to $\mathbf{\Gamma_a}, \mathbf{\Gamma_b}, \bar{\mathbf{a}}, \bar{\mathbf{b}}$.

4) Assume we take a measurement $\mathbf{y} = 3\mathbf{c} + \beta$, where β is centered Gaussian vector independent of $\mathbf{c}$ and with covariance $\mathbf{\Gamma}_\beta$. Using a Kalman filter, give an expression for the updated covariance matrix $\mathbf{\Gamma_c^{up}}$ for $\mathbf{c}$ and the new value for its expectation $\hat{\mathbf{c}}^{\text{up}}$.

Exercise 2. Let us consider the pendulum below. The input of this system is the torque u exerted on the pendulum.



Its state representation is assumed to be :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1 - x_2 + \sin t \end{pmatrix}$$

This is a normalized model in which the coefficients (mass, gravity, length) have all been set to 1.

We assume that we have a gyro placed on the axis of the pendulum that gives us a measurement of x_2 every $\delta = 0.01$ seconds, with a white Gaussian error of standard deviation 0.1rad/sec. We want to reconstruct the state $\mathbf{x}$ of the pendulum when it oscillates near to its stable equilibrium using a Kalman filter. At time $t = 0$,we know that we have exactly $\mathbf{x} = (0.1, 0)$.

1) Linearize around the point $\mathbf{x} = (0, 0)$ and discretize with a sampling period of δ.

2) Find the arguments $(y_k, u_k, \mathbf{\Gamma}_{\alpha_k}, \Gamma_{\beta_k}, \mathbf{A}_k, \mathbf{C}_k)$ to give the Kalman at each iteration k so that the latter generates an estimation $\hat{\mathbf{x}}$ of our state.

3) Explain how the Kalman filter should be initialized.

Correction de l'examen du 10 janvier 2020

Exercise 1.

1) We have

$$\mathbf{\Gamma_v} = \begin{pmatrix} \mathbf{\Gamma_a} & 0 \\ 0 & \mathbf{\Gamma_b} \end{pmatrix}$$

2) Since $\mathbf{a}, \mathbf{b}$ are independent, we have

$$\mathbf{\Gamma_c} = \mathbf{\Gamma_a} + 4\mathbf{\Gamma_b}$$

and

$$\bar{\mathbf{c}} = \bar{\mathbf{a}} + 2\bar{\mathbf{b}}$$

3) We have

$$\pi_{\mathbf{c}}(\mathbf{c}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Gamma_c})}} \cdot \exp\left(-\frac{1}{2}(\mathbf{c}-\bar{\mathbf{c}})^{\mathrm{T}} \cdot \mathbf{\Gamma_c}^{-1} \cdot (\mathbf{c}-\bar{\mathbf{c}})\right)$$

4) From the equations of the Kalman filter, we get

$$\begin{array}{lll} \mathbf{C} & = & 3\mathbf{I} \\ \mathbf{S} & = & \mathbf{C}\cdot\mathbf{\Gamma_c}\cdot\mathbf{C}^{\mathrm{T}} + \mathbf{\Gamma}_\beta \\ \mathbf{K} & = & \mathbf{\Gamma_c}\cdot\mathbf{C}^{\mathrm{T}}\cdot\mathbf{S}^{-1} \\ \widetilde{\mathbf{y}} & = & \mathbf{y} - \mathbf{C}\cdot\bar{\mathbf{c}} \\ \hat{\mathbf{c}}^{\mathrm{up}} & = & \bar{\mathbf{c}} + \mathbf{K}\cdot\widetilde{\mathbf{y}} \\ \mathbf{\Gamma}_{\mathbf{c}}^{\mathrm{up}} & = & (\mathbf{I} - \mathbf{K}\cdot\mathbf{C})\,\mathbf{\Gamma_c} \end{array}$$

Finally, we get

$$\begin{array}{lll} \mathbf{K} & = & \mathbf{\Gamma_c}\cdot 3\cdot(9\cdot\mathbf{\Gamma_c} + \mathbf{\Gamma}_\beta)^{-1} \\ \hat{\mathbf{c}}^{\mathrm{up}} & = & \bar{\mathbf{c}} + \mathbf{K}\cdot(\mathbf{y} - 3\cdot\bar{\mathbf{c}}) \\ \mathbf{\Gamma}^{\mathrm{up}} & = & (\mathbf{I} - 3\mathbf{K})\,\mathbf{\Gamma_c} \end{array}$$

Exercise 2.

1) Linearizing the pendulum yields:

$$\begin{array}{lll} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} & = & \begin{pmatrix} x_2 \\ -\sin x_1 - x_2 + u \end{pmatrix} \simeq \begin{pmatrix} x_2 \\ -x_1 - x_2 + u \end{pmatrix} \\ y & = & x_2 \end{array}$$

where $u(t) = \sin t$.

2) We can tell the Kalman filter that the system it is observing is described by the equations :

$$\begin{array}{l} \mathbf{x}_{k+1} = \underbrace{\begin{pmatrix} 1 & \delta \\ -\delta & 1-\delta \end{pmatrix}}_{\mathbf{A}_k} \mathbf{x}_k + \underbrace{\begin{pmatrix} 0 \\ \delta \end{pmatrix}}_{\mathbf{B}_k} u_k + \alpha_k \\ y_k = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\mathbf{C}_k} \mathbf{x}_k + \beta_k \end{array}$$

where $u_k = \sin \delta k$.

Covariance of the noise. For the covariance matrices of the state noise α_k, we need to take something in the order of δ that depends on the prediction precision (external perturbation such as the wind, approximation due to linearization, negligible friction, etc.). For instance we can take

$$\mathbf{\Gamma}_\alpha = \begin{pmatrix} 0.1\delta & 0 \\ 0 & 0.1\delta \end{pmatrix},$$

Due to the sensor noise, we can take

$$\Gamma_\beta = 0.1^2 = 0.01.$$

3) Initialization. We take $\hat{\mathbf{x}}(0) = (0.1, 0)$ and $\mathbf{\Gamma}_x(0) = \mathbf{0}_{2\times 2}$.