Examen Inertiel, 2A Rob.

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Monday 22 November 2021. Appareils électroniques et polycopié interdits.

Notes manuscrites autorisés

Durée: 1h30.

Exercise 1 (*Interpolation in SO3*). We have a rigid body with an orientation the rotation matrix $\mathbf{R}(t)$. Find an expression for $\mathbf{R}(t)$ such that $\mathbf{R}(0) = \mathbf{R}_0$, $\mathbf{R}(5) = \mathbf{R}_5$, where $\mathbf{R}_0, \mathbf{R}_5$ are two rotation matrices. The expression for $\mathbf{R}(t)$ should depend on $\mathbf{R}_0, \mathbf{R}_5, t$.

Exercise 2 (*Dark side of the moon*). We consider a moon $\mathcal{M}$ turning around a planet $\mathcal{P}$ which is assumed to be static at the origin $\mathbf{0}$. The moon has a mass m which is small compared to the mass of $\mathcal{P}$. The inertia matrix of $\mathcal{M}$ is $\mathbf{I}$. Not only $\mathcal{M}$ rotates around the $\mathcal{P}$, but it also has its own self rotation. The position of the center $\mathcal{M}$ is $\mathbf{p}$ in the world frame, $\mathbf{R}$ is the orientation matrix of $\mathcal{M}$, $\mathbf{v}_r$ is the speed of $\mathcal{M}$ and $\boldsymbol{\omega}_r$ is the rotation vector of $\mathcal{M}$. Both $\mathbf{v}_r$ iand $\boldsymbol{\omega}_r$ are expresses in the moon frame. The gravity generated by $\mathcal{P}$ at $\mathbf{p}$ is assumed to be

$$\mathbf{g} = -\lambda \frac{\mathbf{p}}{\|\mathbf{p}\|^3}$$

where $\lambda > 0$.

1) Give the state equation describing the motion of the moon.

2) Explain how such a system should be simulated.

3) From the state equation of $\mathcal{M}$, compute the derivative of the angular momentum $\mathcal{L} = \mathbf{R} \cdot \mathbf{I} \cdot \boldsymbol{\omega}_r$ of $\mathcal{M}$.

4) Does this explain the fact that our moon has a dark side ?

Correction, Examen Inertiel, 2A Rob
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Exercise 1 (*Interpolation in SO3*). We have a rigid body with an orientation the rotation matrix $\mathbf{R}(t)$. Find an expression for $\mathbf{R}(t)$ such that $\mathbf{R}(0) = \mathbf{R}_0$, $\mathbf{R}(5) = \mathbf{R}_5$, where $\mathbf{R}_0, \mathbf{R}_5$ are two rotation matrices. The expression for $\mathbf{R}(t)$ should depend on $\mathbf{R}_0, \mathbf{R}_5, t$.

Solution.
We take

$$\boldsymbol{\omega} = \frac{1}{5} \cdot \text{Log}(\mathbf{R}_5 \mathbf{R}_0^{\text{T}})$$

Therefore

$$\mathbf{R}(t) = \text{Exp}(t\boldsymbol{\omega}) \cdot \mathbf{R}_0$$

or equivalently

$$\mathbf{R}(t) = \text{Exp}\left(\frac{t}{5} \cdot \text{Log}(\mathbf{R}_5 \mathbf{R}_0^{\text{T}})\right) \cdot \mathbf{R}_0.$$

Exercise 2 (*Dark side of the moon*). We consider a moon $\mathcal{M}$ turning around a planet $\mathcal{P}$ which is assumed to be static at the origin $\mathbf{0}$. The moon has a mass m which is small compared to the mass of $\mathcal{P}$. The inertia matrix of $\mathcal{M}$ is $\mathbf{I}$. Not only $\mathcal{M}$ rotates around the $\mathcal{P}$, but it also has its own self rotation. The position of the center $\mathcal{M}$ is $\mathbf{p}$ in the world frame, $\mathbf{R}$ is the orientation matrix of $\mathcal{M}$, $\mathbf{v}_r$ is the speed of $\mathcal{M}$ and $\boldsymbol{\omega}_r$ is the rotation vector of $\mathcal{M}$. Both $\mathbf{v}_r$ iand $\boldsymbol{\omega}_r$ are expresses in the moon frame. The gravity generated by $\mathcal{P}$ at $\mathbf{p}$ is assumed to be

$$\mathbf{g} = -\lambda \frac{\mathbf{p}}{\|\mathbf{p}\|^3}$$

where $\lambda > 0$.

1) Give the state equation describing the motion of the moon.

We have

$$\begin{cases} \dot{\mathbf{p}} & = & \mathbf{R} \cdot \mathbf{v}_r \\ \dot{\mathbf{R}} & = & \mathbf{R} \cdot (\boldsymbol{\omega}_r \wedge) \\ \dot{\mathbf{v}}_r & = & \mathbf{R}^{\text{T}} \cdot \left(-\lambda \frac{\mathbf{p}}{\|\mathbf{p}\|^3}\right) - \boldsymbol{\omega}_r \wedge \mathbf{v}_r \\ \dot{\boldsymbol{\omega}}_r & = & \mathbf{I}^{-1} \cdot (-\boldsymbol{\omega}_r \wedge \mathbf{I} \cdot \boldsymbol{\omega}_r) \end{cases}$$

2) Explain how such a system should be simulated.

See the course.

3) From the state equation of $\mathcal{M}$, compute the derivative of the angular momentum $\mathcal{L} = \mathbf{R} \cdot \mathbf{I} \cdot \boldsymbol{\omega}_r$ of $\mathcal{M}$.

We have:

$$\begin{array}{rcl} \dot{\mathcal{L}} & = & \dot{\mathbf{R}}\mathbf{I}\boldsymbol{\omega}_r + \mathbf{R} \underbrace{\cdot \dot{\mathbf{I}} \cdot}_{=\mathbf{0}} \boldsymbol{\omega}_r + \mathbf{R} \cdot \mathbf{I} \cdot \underbrace{\dot{\boldsymbol{\omega}}_r}_{=-\mathbf{I}^{-1} \cdot (\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r))} \\ & = & \dot{\mathbf{R}}\mathbf{I}\boldsymbol{\omega}_r - \mathbf{R} (\boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r)) \\ & = & \dot{\mathbf{R}}\mathbf{I}\boldsymbol{\omega}_r - \mathbf{R}\mathbf{R}^{\text{T}} \dot{\mathbf{R}}\mathbf{I} \cdot \boldsymbol{\omega}_r \\ & = & \dot{\mathbf{R}}\mathbf{I}\boldsymbol{\omega}_r - \dot{\mathbf{R}}\mathbf{I} \cdot \boldsymbol{\omega}_r = \mathbf{0}. \end{array}$$

4) Does this explains the fact that our moon has a dark side ?

Of course not. From the previous question, the moon should rotate for ever. Probably a long time ago, the moon had a self rotating motion. Due to the tide effect, a part of the angular momentum of the moon is transfered to the Earth, but the global angular momuntum (Earth + moon) remains constant.