Examen Kalman, 2A Rob, Durée: 1h30

Luc Jaulin, ENSTA-Bretagne, Monday December 14 2020.

Appareils électroniques interdits. Polycopié et notes manuscrites autorisés

---

Do not perform any heavy computation. Write expressions instead.

Exercise 1. We consider the random vector $\mathbf{x}$ with probability density function given by

$$\pi_{\mathbf{x}}(\mathbf{x}) = a \cdot \exp\left((\mathbf{x}-\bar{\mathbf{x}})^{\mathrm{T}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \cdot (\mathbf{x}-\bar{\mathbf{x}})\right)$$

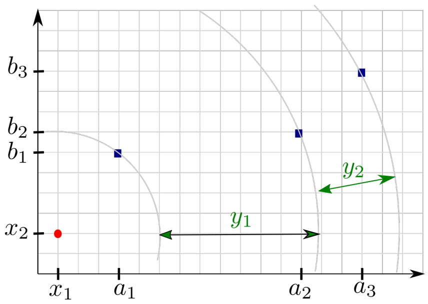
where $\bar{\mathbf{x}} = (1,2)^{\mathrm{T}}$.

1) Give the values of a and $\mathbf{\Gamma_x}$, the covariance matrix of $\mathbf{x}$.

2) Consider the state vector $\mathbf{y} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} \mathbf{x} + \mathbf{c}$ where $\mathbf{c}$ is a centered random vector with covariance matrix equal to identity. Give an expression for the probability density function of $\mathbf{y}$.

Exercise 2. *Pseudorange multilateration* is a technique for determining a robot position based on measurement of the *times of arrival* of signals having a known speed when propagating from the stations to the robot. These stations are at known locations and have synchronized clocks, but these clocks are not synchronized with the clock of the robot. Consequently, the robot do not measure the distance to the stations. Instead, it measures the difference of distances between its position and that of the stations.

Consider one robot at position (x_1, x_2) in the plane where three stations at known positions $(a_i, b_i), i \in \{1,2,3\}$ emit at the same unknown time a sound, as illustrated the Figure. The three corresponding sounds are received at times t_i, as given by the following table. We assume that the measurement errors for the t_i are all Gaussian, independent, with a variance equal to $\sigma_{t_i}^2 = 4$.



i	1	2	3
a_i	4	13	16
b_i	6	7	10
t_i	15	23	27

Since the emission time is not known by the robot (near to the origin in the figure) , we cannot translate these times into distances. However, the difference of times is directly related to difference of distances (called the *pseudo distances*). The pseudo distances are y_1, y_2 in the figure. We assume for simplicity that speed of the sound is $1 m.s^{-1}$.

1) Give the peudo-distance vector $\mathbf{y} = (y_1, y_2)^{\mathrm{T}}$, where y_1 is the pseudo distance between stations 2 and 1 and y_2 is the peudo-distance y_2 between stations 3 and 2. Give also the associated covariance matrix.

2) The robot wants to estimate its position from $\mathbf{y}$. For this, it assumes that it is at point $\hat{\mathbf{x}}_0 = (0,0)^{\mathrm{T}}$ with a covariance matrix equal to $\mathbf{\Gamma}_0 = 10^2 \cdot \mathbf{I}\cdot$ where $\mathbf{I}$ is the identity matrix. Using the the correction part of the extended Kalman filter, write the formulas that provides an estimation $\hat{\mathbf{x}}_1$ of the position of the robot. Give also the expressions for the covariance matrix $\mathbf{\Gamma}_1$ associated to the localization.

Correction, Examen Kalman, 2A Rob

Monday 7 December 2020

Exercise 1. We consider the random vector $\mathbf{x}$ with probability density function given by

$$\pi_{\mathbf{x}}(\mathbf{x}) = a \cdot \exp\left((\mathbf{x}-\bar{\mathbf{x}})^{\mathrm{T}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \cdot (\mathbf{x}-\bar{\mathbf{x}})\right)$$

where $\bar{\mathbf{x}} = (1,2)^{\mathrm{T}}$.

1) Give the values of a and $\mathbf{\Gamma_x}$, the covariance matrix of $\mathbf{x}$.

Solution. We have

$$\pi_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{\Gamma_x})}} \cdot \exp\left(-\frac{1}{2}(\mathbf{x}-\bar{\mathbf{x}})^{\mathrm{T}} \cdot \mathbf{\Gamma_x}^{-1} \cdot (\mathbf{x}-\bar{\mathbf{x}})\right).$$

By identification, we get

$$-\frac{1}{2} \cdot \mathbf{\Gamma_x}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

i.e,

$$\mathbf{\Gamma_x} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

and

$$a = \frac{1}{\sqrt{(2\pi)^2 \frac{1}{8}}} = \frac{\sqrt{2}}{\pi}$$

2) Consider the state vector $\mathbf{y} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} \mathbf{x} + \mathbf{c}$ where $\mathbf{c}$ is a centered random vector with covariance matrix equal to identity.

Give an expression for the probability density function of $\mathbf{y}$.

2) Using the prediction step of the Kalman filter,

$$\mathbf{\Gamma_y} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ -1 & 1 \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & \frac{7}{4} \end{pmatrix}$$

$$\bar{\mathbf{y}} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ -1 & 1 \end{pmatrix} \bar{\mathbf{x}} = \begin{pmatrix} 5 \\ 8 \\ 1 \end{pmatrix}.$$

Thus

$$\frac{1}{\sqrt{(2\pi)^n \det \begin{pmatrix} \frac{5}{2} & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & \frac{7}{4} \end{pmatrix}}} \cdot \exp\left(-\frac{1}{2}\left(\mathbf{y} - \begin{pmatrix} 5 \\ 8 \\ 1 \end{pmatrix}\right)^{\mathrm{T}} \cdot \begin{pmatrix} \frac{5}{2} & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & \frac{7}{4} \end{pmatrix}^{-1} \cdot \left(\mathbf{y} - \begin{pmatrix} 5 \\ 8 \\ 1 \end{pmatrix}\right)\right)$$

Exercise 2.

1) Give the peudo-distance vector $y=(y_1,y_2)^{\mathrm{T}}$, where y_1 is the pseudo distance between stations 2 and 1 and y_2 is the peudo-distance y_2 between stations 3 and 2. Give also the associated covariance matrix.

Solution. The measured pseudo-distance vector is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

The associate covariance matrix is

$$\mathbf{\Gamma}_{\boldsymbol{\beta}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix}$$

2) The robot wants to estimate its position from the pseudo distances. For this, it assumes that it is at point $\hat{\mathbf{x}}_0=(0,0)^{\mathrm{T}}$ with a covariance matrix equal to $\mathbf{\Gamma}_0 = 10^2 \cdot \mathbf{I}\cdot$ where $\mathbf{I}$ is the identity matrix. Using the the correction part of the extended Kalman filter, write the formulas that provides an estimation $\hat{\mathbf{x}}_1$ of the position of the robot. Give also the covariance matrix $\mathbf{\Gamma}_1$ associated to the localization.

Solution.

We have

$$\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{(x_1-13)^2+(x_2-7)^2} - \sqrt{(x_1-4)^2+(x_2-6)^2} \\ \sqrt{(x_1-16)^2+(x_2-10)^2} - \sqrt{(x_1-13)^2+(x_2-7)^2} \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$\frac{d\mathbf{g}}{d\mathbf{x}}(\mathbf{0}) = \begin{pmatrix} \frac{13}{\sqrt{13^2+7^2}} - \frac{4}{\sqrt{4^2+6^2}} & \frac{7}{\sqrt{13^2+7^2}} - \frac{6}{\sqrt{4^2+6^2}} \\ \frac{16}{\sqrt{16^2+10^2}} - \frac{13}{\sqrt{13^2+7^2}} & \frac{10}{\sqrt{16^2+10^2}} - \frac{7}{\sqrt{13^2+7^2}} \end{pmatrix} = \begin{pmatrix} 0.33 & -0.36 \\ -0.03 & 0.06 \end{pmatrix}$$

We take the observation equations of the extended Kalman filter with $\hat{\mathbf{x}}_{k|k-1}=\mathbf{0}$,$\hat{\mathbf{x}}_{k|k}=\hat{\mathbf{x}}$,$\mathbf{\Gamma}_{k|k-1}=\mathbf{\Gamma}_0$, $\mathbf{\Gamma}_{k|k-1}=\mathbf{\Gamma}_0$. We get

$$\begin{aligned} \mathbf{C} &= \tfrac{d\mathbf{g}}{d\mathbf{x}}(\mathbf{0}) = \begin{pmatrix} 0.33 & -0.36 \\ -0.03 & 0.06 \end{pmatrix} \\ \hat{\mathbf{x}}_1 &= \mathbf{K}\cdot(\mathbf{y}-\mathbf{g}(\mathbf{0})) \\ \mathbf{\Gamma}_1 &= (\mathbf{I}-\mathbf{KC})\,10^2 \\ \mathbf{K} &= 10^2\mathbf{C}^{\mathrm{T}}(10^2\mathbf{C}\mathbf{C}^{\mathrm{T}}+\mathbf{\Gamma}_{\boldsymbol{\beta}})^{-1} \end{aligned}$$