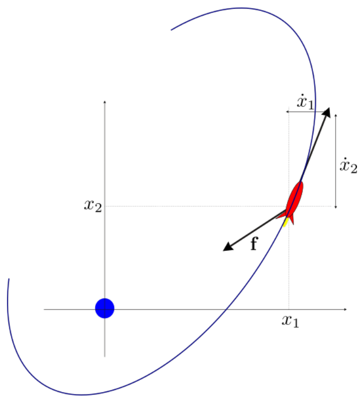
Examen guidage, 2A Rob.
Luc Jaulin, ENSTA-Bretagne
2022, april Monday 25
Appareils électroniques et polycopié interdits.
Notes manuscrites autorisées
Durée: 1h30.

Exercise 1 (*Satellite rocket*). Consider the rocket, with coordinates (x_1, x_2) in orbit around a planet. The density of the surrounding atmosphere is neglected but is supposed to be significant enough to orientate the rocket forward.



Planet (blue) at coordinates $(0, 0)$ and the satellite (red) at coordinates (x_1, x_2)

1) We assume that we can control the tangent acceleration of the satellite by a variable u. Show that a possible normalized state-space model for the system is

$$\begin{cases} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{x_1}{\sqrt{x_1^2+x_2^2}^3} + x_3 u \\ \dot{x}_4 &= -\frac{x_2}{\sqrt{x_1^2+x_2^2}^3} + x_4 u \end{cases}$$

2) Propose a feedback linearization based method to find a controller for $u(\mathbf{x})$ such that the rocket converges to the circle of radius R. For this, we will choose as an output $y = x_1^2 + x_2^2 - R^2$. Conclude.

Exercise 2 (*Char robot*). Consider a robot moving on a plane and described by the following state equations:

$$\begin{cases} \dot{x}_1 &= \cos x_3 \\ \dot{x}_2 &= \sin x_3 \\ \dot{x}_3 &= u \end{cases}$$

where x_3 is the heading of the robot and (x_1, x_2) are the coordinates of its center. The state vector is given by $\mathbf{x} = (x_1, x_2, x_3)$.

1) Find the expression of a two-dimensional vector field $\mathbf{f}(x_1, x_2)$ such that the limit cycle is clockwise stable and corresponds to an ellipse with a major radius 2, a minor radius 1 and an inclination of $\frac{\pi}{4}$. Draw the corresponding vector field.

2) Find a controller so that the position of our robot follows the vector field $\mathbf{f}(x_1, x_2)$.

Correction, guidage, 2A Rob
2022, april Monday 25

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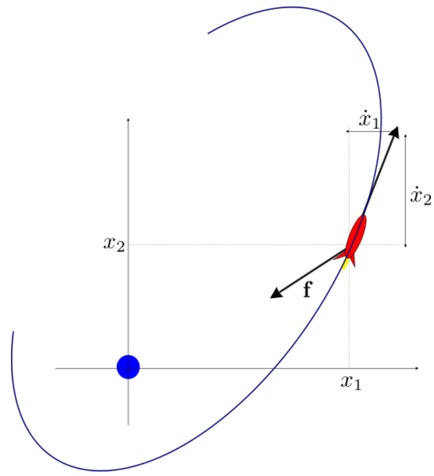

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1) We assume that we can control the tangent acceleration of the satellite by a variable u. Show that a possible normalized state-space model for the system is

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Solution. 1) Newton's universal law of gravitation states that the earth exerts a gravitational force **f** of attraction on the rocket. The direction of **f** is along the unit vector **u** directed by the line joining the two objects. The magnitude of the force is proportional to the product of the gravitational masses of the objects, and inversely proportional to the square of the distance between them. We have

$$\mathbf{f} = -G\frac{Mm}{x_1^2 + x_2^2}\mathbf{u},$$

where M is the mass of the earth, m is the mass of the rocket and $G = 6.67x10^{-11} Nm^2kg^{-2}$ is the Newton's constant. We have

$$\mathbf{f} = -G\frac{Mm}{\sqrt{x_1^2 + x_2^2}^3}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Now, from the Newton's second law, we have

$$\mathbf{f} = m\frac{d\mathbf{v}}{dt} = m\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix},$$

where **v** is the velocity of the rocket. Thus

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -G\frac{M}{\sqrt{x_1^2 + x_2^2}^3}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If we set $\alpha = GM = 1$, and if we take into account the possibility to control only the tangent acceleration, we get the following state equations:

$$\begin{cases} \dot{x}_1 & = & x_3 \\ \dot{x}_2 & = & x_4 \\ \dot{x}_3 & = & -\frac{x_1}{\sqrt{x_1^2+x_2^2}^3} + x_3 u \\ \dot{x}_4 & = & -\frac{x_2}{\sqrt{x_1^2+x_2^2}^3} + x_4 u \end{cases}$$

2) Propose a feedback linearization based method to find a controller for $u(\mathbf{x})$ such that the rocket converges to the circle of radius R. For this, we will choose as an output $y = x_1^2 + x_2^2 - R^2$. Conclude.

Solution. To apply a feedback linearization method, take as an output $y = x_1^2 + x_2^2 - R^2$. We have

$$\begin{array}{rcl} y & = & x_1^2 + x_2^2 - R^2 \\ \dot{y} & = & 2x_1x_3 + 2x_2x_4 \\ \ddot{y} & = & 2\dot{x}_1x_3 + 2x_1\dot{x}_3 + 2\dot{x}_2x_4 + 2x_2\dot{x}_4 \\ & = & 2x_3^2 + 2x_1\left(-\frac{x_1}{\sqrt{x_1^2+x_2^2}^3} + x_3u\right) + 2x_4^2 + 2x_2\left(-\frac{x_2}{\sqrt{x_1^2+x_2^2}^3} + x_4u\right) \\ & = & 2\left(x_3^2 + x_4^2\right) - \frac{2}{\sqrt{x_1^2+x_2^2}} + \underbrace{(2x_1x_3 + 2x_2x_4)}_{a(\mathbf{x})}u \end{array} \quad (1)$$

A feedback linearization generates a singularity for $a(\mathbf{x}) = 2x_1x_3 + 2x_2x_4 = 0$ which corresponds to $\dot{y} = 0$. Now, if we want to have $y \to 0$, we will also have $\dot{y} \to 0$ and we will thus converge to the singular surface $a(\mathbf{x}) = 0$. This may be a problem near the objective.

Exercise 2 (*Char robot*). Consider a robot moving on a plane and described by the following state equations:

$$\begin{cases} \dot{x}_1 & = & \cos x_3 \\ \dot{x}_2 & = & \sin x_3 \\ \dot{x}_3 & = & u \end{cases}$$

where x_3 is the heading of the robot and (x_1, x_2) are the coordinates of its center. The state vector is given by $\mathbf{x} = (x_1, x_2, x_3)$.

1) Find the expression of a two-dimensional vector field $\mathbf{f}(x_1, x_2)$ such that the limit cycle is clockwise stable and corresponds to an ellipse with a major radius 2, a minor radius 1 and an inclination of $\frac{\pi}{4}$. Draw the corresponding vector field.

Solution

We take build a field corresponding to a rotating attractive circle of radius 1. For this, we add two fields: one corresponding to an attractive circle and one corresponding to a rotation field:

$$\mathbf{c}(\mathbf{x}) = \underbrace{-(x_1^2 + x_2^2 - 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{attractive circle}} + \underbrace{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}}_{\text{rotation}} = \begin{pmatrix} -x_1^3 - x_1x_2^2 + x_1 - x_2 \\ -x_2^3 - x_1^2x_2 + x_1 + x_2 \end{pmatrix}$$

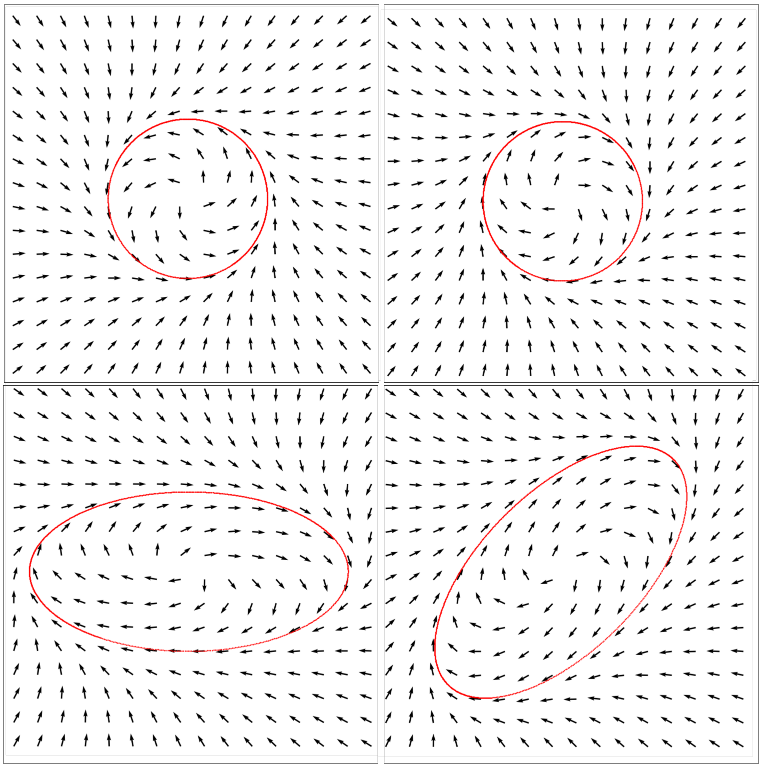
The required vector field is given by

$$\mathbf{f}(\mathbf{x}) = \mathbf{A} \cdot \mathbf{c}(\mathbf{A}^{-1} \cdot \mathbf{x})$$

where

$$\mathbf{A} = \underbrace{\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}}_{\mathbf{R}} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{D}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathbf{S}}$$

The first matrix **S** changes the orientation of the field to get a clockwise rotation of the trajectories, **D** gives the right dimensions of the limit ellipse and **R** rotates the field to get the right inclination. The figure depicts the action of **S**,**D**,**R**, respectively.



2) Find a controller so that the position of our robot follows the vector field $\mathbf{f}(x_1, x_2)$.

Solution. We take

$$u = \text{sawtooth}\left(\text{atan2}(\mathbf{f}(x_1, x_2)) - x_3\right)$$