Examen Kalman, 2A Rob
Luc Jaulin, ENSTA-Bretagne, Mardi 13 décembre 2022, 10h20-11h50.
Appareils électroniques interdits. Polycopié interdit. Notes manuscrites autorisées

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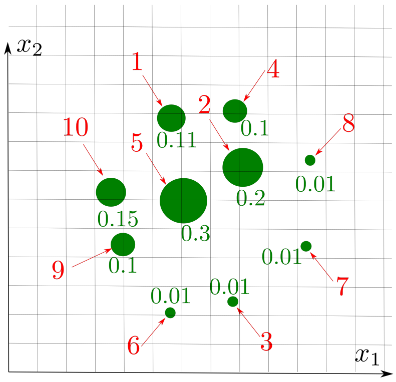
Exercice 1. *Resampling*

On considère le nuage de particules de la figure représentant une densité de probabilité $\pi_{\mathbf{x}}$ pour la position $\mathbf{x} = (x_1, x_2)$ d'un robot. A chaque particule j est associé un poids w_j.

1) Donner une formule en fonction des $x_1(j)$, $x_2(j)$ et des w_j pour approximer l'espérance mathématique $E(\mathbf{x})$ et la matrice de covariance $\Gamma_{\mathbf{x}}$ de $\mathbf{x}$.

2) On voudrait rééchantillonner le nuage de façon à approximer la même densité de probabilité, mais cette fois, tous les poids doivent être égaux.

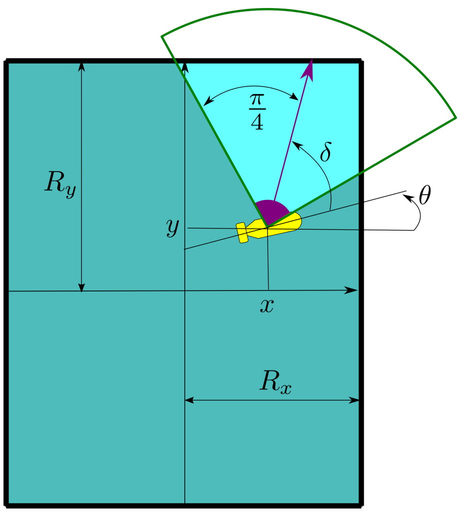
j	1	2	3	4	5	6	7	8	9	10
w_j	0.11	0.2	0.01	0.1	0.3	0.01	0.01	0.01	0.1	0.15



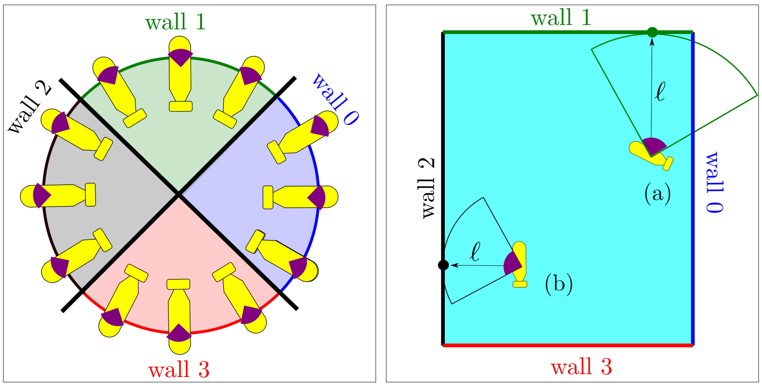
Donner et dessiner le nouveau nuage de particules. Justifier vos calculs.

Exercice 2. *Localisation d'un robot dans une piscine*

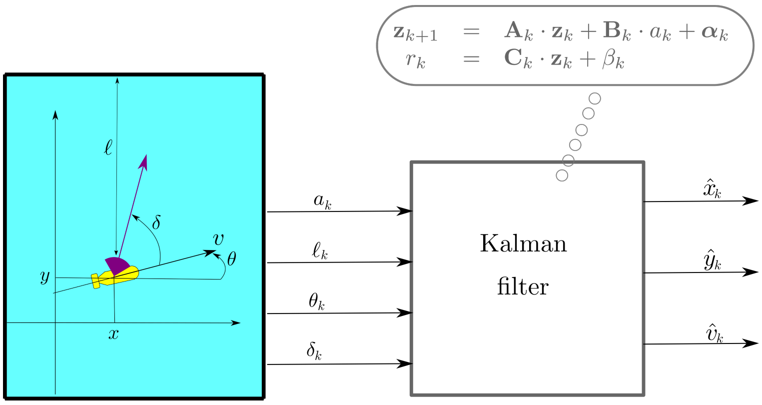
Consider an underwater robot moving at a constant depth within a rectangular pool of length $2R_y$ and width $2R_x$. A sonar rotates with a constant angular speed $\dot{\delta}$. We want to estimate the coordinates (x, y) of the robot. The origin of the coordinate system is in middle of the pool. We assume that the angle δ of the sonar and the heading angle θ are known. The sonar illuminates the environment inside an emission cone with an angle $\pm\frac{\pi}{4}$, as illustrated by the figure.



The tangential acceleration a is measured with an accelerometer. Every $dt = 0.1$s, the sonar returns the distance ℓ to the wall in front of it. As illustrated by the figure below the number w of the wall involved in the distance measurement (called the *hit wall*) only depends on the angle $\theta + \delta$ and not on the position of the robot. In the configuration (a), the sonar in on the right of the robot and since $\theta + \delta \simeq \frac{\pi}{2}$, Wall 1 is hit. In configuration (b), the sonar is on the left of the robot and since $\theta + \delta \simeq \pi$, Wall 2 is hit.



We want the build a Kalman filter to estimate the position (x, y) of the robot from the measurements $a_k, \ell_k, \theta_k, \delta_k$ at time $t_k = dt \cdot k$.



The Kalman filter will be built on the following kinematic model

$$\begin{cases} \dot{x} &= v \cdot \cos\theta \\ \dot{y} &= v \cdot \sin\theta \\ \dot{v} &= a \end{cases}$$

The state vector is $\mathbf{z} = (x, y, v)$ and the input is a. The Kalman filter assumes the following linear state equations

$$\begin{aligned} \mathbf{z}_{k+1} &= \mathbf{A}_k \cdot \mathbf{z}_k + \mathbf{B}_k \cdot a_k + \boldsymbol{\alpha}_k \\ r_k &= \mathbf{C}_k \cdot \mathbf{z}_k + \beta_k \end{aligned}$$

where $\boldsymbol{\alpha}_k$ and β_k are white Gaussian noises.

1) Give the expressions for $\mathbf{A}_k, \mathbf{B}_k$ and $\mathbf{C}_k$ we should take to have the estimations $\hat{x}$, $\hat{y}$, $\hat{v}$ of the variables x, y, v.

2) Explain how the quantity $r(k)$ should be chosen from the measurements.

Correction, Examen Kalman, 2A Rob
Mardi 13 décembre 2022, 10h20-11h50

Correction of Exercise 1. (*resampling*)

1) On a

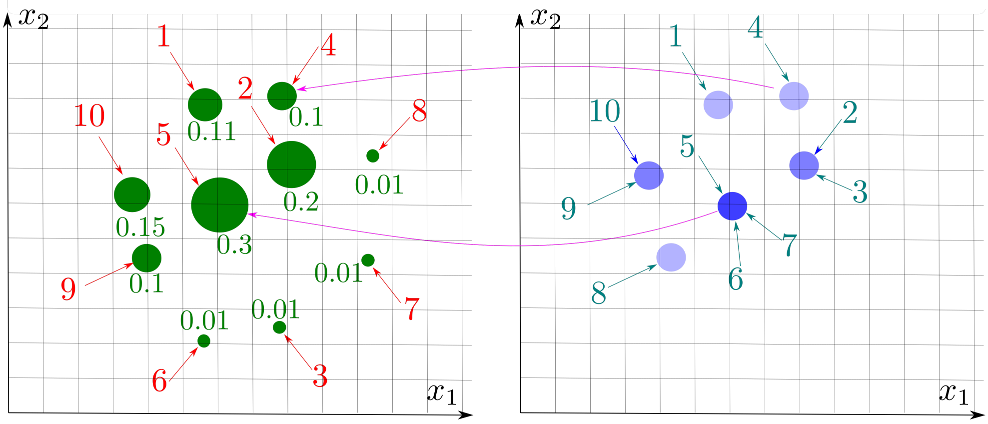
$$\bar{\mathbf{x}} = \begin{pmatrix} \sum_j w_j x_1(j) \\ \sum_j w_j x_2(j) \end{pmatrix}$$

et

$$\Gamma_{\mathbf{x}} = \begin{pmatrix} \sum_j w_j (x_1(j) - \bar{x}_1)^2 & \sum_j w_j (x_1(j) - \bar{x}_1) \cdot (x_2(j) - \bar{x}_2) \\ \sum_j w_j (x_1(j) - \bar{x}_1) \cdot (x_2(j) - \bar{x}_2) & \sum_j w_j (x_2(j) - \bar{x}_2)^2 \end{pmatrix}$$

2) The principle is to consider particles as a container containing a liquid corresponding to the w_ℓ's. The new particles are initially empty and all the fluid has to be poured from the old particles to the new one. We bring the new particles one by one and we fill them with the fluid until the new particle contains $1/N$. As soon as an old particle is empty, we delete it. When a new particle contains $\frac{1}{N}$ we drop it at the place where the last old particle was. This is illustrated by the figure where 10 green particles (green) are resampled into the 10 blues particles. Each blue particle is associated to one green particle. When two blue particles overlap, the blue disk is darker. The 3 blue particles 5,6,7 which overlap are associated to a unique green particle. The result of the process is an index J which tells that the ith new particle has the position of the $J(i)'s$ old particle. To compute the index $J(i)$ we synchronize the cumulative sum $\gamma = \sum_{\ell \le j} w_\ell$ for $\mathcal{W}$ and the cumulative sum $\sum_{\ell \le i} w'_\ell = \frac{i}{N}$ for $\mathcal{W}'$.

j	1	2	3	4	5	6	7	8	9	10
w_j	0.11	0.2	0.01	0.1	0.3	0.01	0.01	0.01	0.1	0.15
$\sum_{\ell \le j} w_\ell$	0.11	0.31	0.32	0.42	0.72	0.73	0.74	0.75	0.85	1
$\sum_{\ell \le i} w'_\ell$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$J(i)$	1	2	2	4	5	5	5	9	10	10
i	1	2	3	4	5	6	7	8	9	10



Resampling: generation of 10 particles (blue) from 10 particles (green). Several blue particles may overlap. The probability for a particle with a low weight to survive after a resampling is small.

This can be computed by the following algorithm

Function RESAMPLE $(\mathcal{W})$

$i = 1$; $j = 1$;
$\gamma = w_1$
while $i \leq N$
    if $\frac{i}{N} < \gamma$ then $J(i) = j$; $i = i + 1$
    else $j = j + 1$; $\gamma = \gamma + w_j$
Return J

Correction of Exercise 2 (robot localization in a pool)

We take

$$\begin{array}{rcl} \mathbf{z}_{k+1} & = & \begin{pmatrix} 1 & 0 & \cos\theta_k \cdot dt \\ 0 & 1 & \sin\theta_k \cdot dt \\ 0 & 0 & 1 \end{pmatrix} \cdot \mathbf{z}_k + \begin{pmatrix} 0 \\ 0 \\ dt \end{pmatrix} a_k \\ r_k & = & \mathbf{C}_k \cdot \mathbf{z}_k \end{array}$$

To build $\mathbf{C}_k$, we need to know which is the hit wall. Its number is

$$w = \text{modulo}(\text{round}(\frac{2}{\pi}(\theta + \delta)), 4)$$

Depending on w, the output r_k and the matrix $\mathbf{C}_k$ are given by the following table

	r_k	$\mathbf{C}_k$
$w = 0$	$R_x - \ell$	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$w = 1$	$R_y - \ell$	$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$
$w = 2$	$\ell - R_x$	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
$w = 3$	$\ell - R_y$	$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$

For instance if $w = 0$, we have

$$\underbrace{r_k}_{=R_x - \ell} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \mathbf{z}_k}_{=x_k} + \beta_k$$

i.e. $x_k \simeq R_x - \ell$. This is consistent with the fact that when the sonar hit Wall 0 or Wall 2, we measure indirectly x_k and when the sonar hit Wall 1 or Wall 3, we measure indirectly y_k.

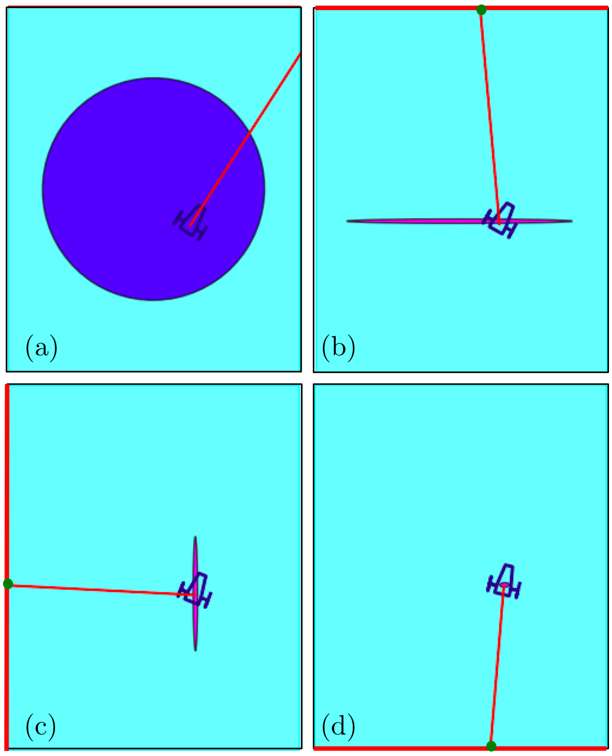
For initialization of the Kalman filter, we took

$$\hat{\mathbf{z}}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{\Gamma}_{\mathbf{z}} = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix}$$

We got the results illustrated by the figure below. We have taken the covariance matrices for the noises $\boldsymbol{\alpha}$,β as

$$\mathbf{\Gamma}_{\boldsymbol{\alpha}} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix} \text{ and } \mathbf{\Gamma}_\beta = 0.04$$

and generated the noises accordingly. Subfigure (a) corresponds to the initialization. In Subfigure (b) Wall 1 is hit. The confidence ellipse for (z_1, z_2) (or equivalently for (x, y) becomes flat in y. In Subfigure (c) Wall 2 is hit. The confidence ellipse becomes flat in x. The added uncertainty in x is due to the fact that the speed v is still uncertain. In Subfigure (d) Wall 3 is hit. The ellipse becomes is a small both x, y. We can also check that the confidence interval for z_3 (which corresponds to v) is small also.


(a)
(b)
(c)
(d)