Examen Inertiel, ENSI 2.
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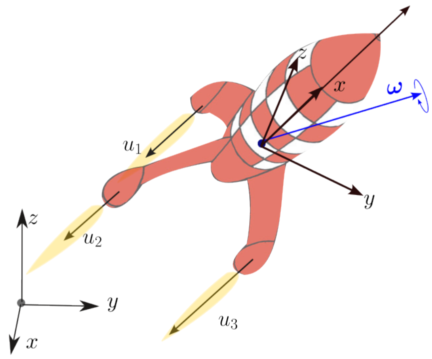
Le vendredi 15 décembre 2023. Appareils électroniques interdits.
Notes manuscrites autorisées
Durée: 75 minutes.

We consider the moon rocket of the figure



The inertia matrix $\mathbf{I}$, the mass m are assumed to be constant. The state is composed of

- the orientation matrix $\mathbf{R}$, the position $\mathbf{p}$ (both in the world frame),
- the velocity $\mathbf{v}_r$ and the rotation vector $\boldsymbol{\omega}_r$ (both expressed in the rocket frame).

The rocket has 3 propellers parallel to the x axis of the rocket. The ith propeller is at position $\mathbf{q}(i)$ and produces a force u_i in the direction $\mathbf{d}(i)$. We have $\mathbf{d}(i) = (1, 0, 0)^{\mathrm{T}}$ (in the rocket frame) for all $i \in \{1, 2, 3\}$. In the rocket frame, the coordinates of $\mathbf{q}(i)$ are given by:

i	1	2	3
$\mathbf{q}(i)$	$\begin{pmatrix} -10 \\ 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -10 \\ -1 \\ \sqrt{3} \end{pmatrix}$	$\begin{pmatrix} -10 \\ -1 \\ -\sqrt{3} \end{pmatrix}$

Each force u_i contributes to the resultant as $\mathbf{d}(i) \cdot u_i$ and to the torque as $\mathbf{q}(i) \wedge \mathbf{d}(i) \cdot u_i$.

1) Give the state equations of the system.

2) Due to a (non destructive) collision with a small meteor, the rocket now rotates around an undesired axis with some unconfortable precession.

Propose a controller so that the rotation vector of the rocket becomes parallel to the x-axis. You should give an expression of $\mathbf{u}$ with respect to $\boldsymbol{\omega}_r = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$.

Hint. To achieve our goal, we will cancel ω_2 and ω_3 by taking:

$$\begin{cases} I_2 \dot{\omega}_2 + \omega_2 & = & 0 \\ I_3 \dot{\omega}_3 + \omega_3 & = & 0 \end{cases}$$

Correction de l'examen d'inertiel du 15 décembre 2023

1) The resulting force in the body frame is

$$\mathbf{f}_r = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

and the torque is:

$$\begin{aligned} \boldsymbol{\tau}_r &= \begin{pmatrix} \mathbf{q}(1) \wedge \mathbf{d}(1) & \mathbf{q}(2) \wedge \mathbf{d}(2) & \mathbf{q}(3) \wedge \mathbf{d}(3) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3} & -\sqrt{3} \\ -2 & 1 & 1 \end{pmatrix}}_{\mathbf{C}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \end{aligned}$$

Indeed,

$$\begin{aligned} \begin{pmatrix} -10 \\ 2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \\ \begin{pmatrix} -10 \\ -1 \\ \sqrt{3} \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ \sqrt{3} \\ 1 \end{pmatrix} \\ \begin{pmatrix} -10 \\ -1 \\ -\sqrt{3} \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ -\sqrt{3} \\ 1 \end{pmatrix} \end{aligned}$$

Therefore, we get that the state equations:

$$\begin{cases} \dot{\mathbf{p}} &= \mathbf{R} \cdot \mathbf{v}_r \\ \dot{\mathbf{R}} &= \mathbf{R} \cdot (\boldsymbol{\omega}_r \wedge) \\ \dot{\mathbf{v}}_r &= \frac{u_1+u_2+u_3}{m} - \boldsymbol{\omega}_r \wedge \mathbf{v}_r \\ \dot{\boldsymbol{\omega}}_r &= \mathbf{I}^{-1} \cdot (\mathbf{C}\mathbf{u} - \boldsymbol{\omega}_r \wedge (\mathbf{I} \cdot \boldsymbol{\omega}_r)) \end{cases}$$

2) We cannot get any torque along the x axis. We have

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \mathbf{I}^{-1} \cdot \left(\begin{pmatrix} 0 \\ \sqrt{3}u_2 - \sqrt{3}u_3 \\ -2u_1 + u_2 + u_3 \end{pmatrix} - \begin{pmatrix} \frac{I_3-I_2}{I_1}\omega_2\omega_3 \\ \frac{I_1-I_3}{I_2}\omega_3\omega_1 \\ \frac{I_2-I_1}{I_3}\omega_1\omega_2 \end{pmatrix} \right)$$

Since

$$\begin{cases} I_2\dot{\omega}_2 + \omega_2 &= 0 \\ I_3\dot{\omega}_3 + \omega_3 &= 0 \end{cases}$$

we have

$$\begin{cases} -\omega_2 &= \sqrt{3}u_2 - \sqrt{3}u_3 - \frac{I_1-I_3}{I_2}\omega_3\omega_1 \\ -\omega_3 &= -2u_1 + u_2 + u_3 - \frac{I_2-I_1}{I_3}\omega_1\omega_2 \end{cases}$$

Set $u_1 = 0$. We get the controller:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{I_1-I_3}{I_2}\omega_3\omega_1 - \omega_2 \\ \frac{I_2-I_1}{I_3}\omega_1\omega_2 - \omega_3 \end{pmatrix} \end{pmatrix}$$