Examen Inertiel, Rob 2A.
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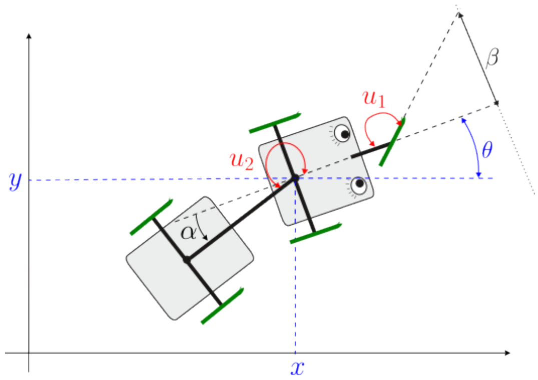
Lundi 9 décembre 2024, 14h-15h30
Appareils électroniques, polycopiés, photocopies, interdits.
Notes manuscrites autorisées

Exercise 1. *Lie groups.* Consider the set of matrices $\mathbb{A}$ of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where a and b belong to $\mathbb{R}$ and are such that $a^2 + b^2 \neq 0$.

1) Show that $\mathbb{A}$ is a Lie group with respect to the multiplication.

2) Define the corresponding Lie algebra.

Exercise 2. *Skate with Lie brackets.* Consider the skating vehicle with five ice skates (or *blades*) which is designed to move on a frozen lake.



This system has two inputs: u_1 which tunes the steering blade orientation represented by β, the tangent of the front blade angle. We have chosen the tangent β in order to avoid the singularities at $\pi/2$. The input u_2 is the torque exerted at the articulation. The thrust therefore only comes from u_2 and recalls the propulsion mode of an eel. The state variables are chosen to be $\mathbf{x} = (x, y, \theta, v, \alpha, \beta)$, where x, y, θ correspond to the position of the front cart, v represents the speed of the center of the front chart axle and α is the angle between the two carts.

1) Show that a possible model is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} v\cos\theta \\ v\sin\theta \\ v\beta \\ 0 \\ -v(\beta+\sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\beta-\sin\alpha \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{g}_2} \cdot u_2$$

2) Compute the Lie brackets $[\mathbf{g}_1, \mathbf{g}_2]$ between $\mathbf{g}_1$ and $\mathbf{g}_2$.

3) Find a controller which allows the robot to move along the direction given by $[\mathbf{g}_1, \mathbf{g}_2]$.

4) Deduce a controller which drives the vehicle to a heading $\bar{\theta}$ and a speed $\bar{v}$.

Correction de l'examen d'inertiel 9 décembre 2024

Exercise 1. *Lie groups.*

1) The matrix

$$z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

represents the complex number $z = a + ib$. It is a matrix representation of $\mathbb{C}$ the set of complex numbers. Note that it was not necessary to know this to answer the questions. We have

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix}$$

It is consistent with the the multiplication of complex numbers

$$(a + ib)(c + id) = ac - bd + i(ad + bc)$$

Moreover

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2 + b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Indeed $(a + ib)(a - ib) = a^2 + b^2$. The identity of the Lie group $\mathbb{A}$ is

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2) The generators have the form

$$\mathbf{E}_1 = \frac{\partial z}{\partial a}(\mathbf{0}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{E}_2 = \frac{\partial z}{\partial b}(\mathbf{0}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The Lie algebra is the set of matrices which as the form

$$\alpha\mathbf{E}_1 + \beta\mathbf{E}_2 = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. The Lie algebra and the Lie group are almost identical (except for zero which belongs to the Lie algebra and not to $\mathbb{A}$).

The exponential exists. For instance

$$\exp\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Euler formula $e^{i\pi} = -1$ reads

$$\exp\begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 2. *Skate with Lie brackets.*

2) We have

$$
\begin{array}{rcl}
[\mathbf{g}_1,\mathbf{g}_2] & = & \frac{d\mathbf{g}_2}{d\mathbf{x}}\cdot\mathbf{g}_1 - \underbrace{\frac{d\mathbf{g}_1}{d\mathbf{x}}}_{=\mathbf{0}}\cdot\mathbf{g}_2 \\
 & = & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\cos\alpha & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}
\end{array}
$$

3) Note first that if we want to control v to be equal to $\bar{v}$, since

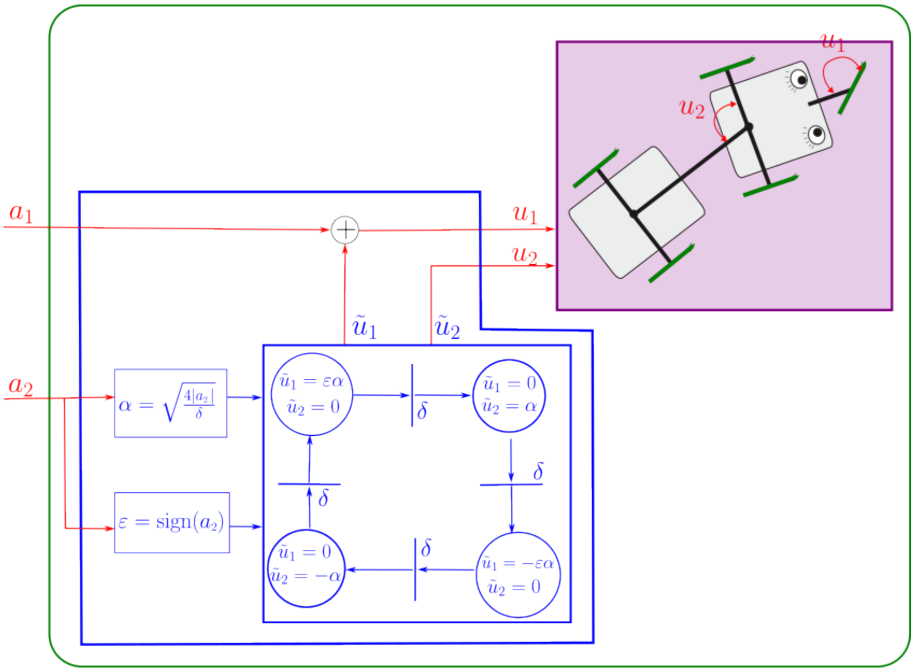
$$\dot{v} \;=\; -(\beta+\sin\delta)\,u_2,$$

we could take

$$u_2 = -\frac{\bar{v}-v}{\beta+\sin\delta}$$

If we do this, we get $\dot{v}=\bar{v}-v$ and thus we should converge to the wanted value for the speed. Now, we quickly reach a singularity when $\beta+\sin\delta=0$. We conclude that this controller is not acceptable. A *biomimetic* controller that imitates the propulsion of the snake or the eel might be feasible.

We observe that $[\mathbf{g}_1,\mathbf{g}_2]$ points toward the direction of v in the state space. To move along $[\mathbf{g}_1,\mathbf{g}_2]$, we take the following controller.



We thus get the state equations

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} v\cos\theta \\ v\sin\theta \\ v\beta \\ 0 \\ -v\,(\beta+\sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1}\cdot a_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{[\mathbf{g}_1,\mathbf{g}_2]}\cdot a_2
$$

To control the speed and the steering angle we select the two equations

$$\begin{pmatrix} \dot{v} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot a_1 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot a_2$$

i.e.

$$\begin{pmatrix} \dot{v} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{a}$$

We want

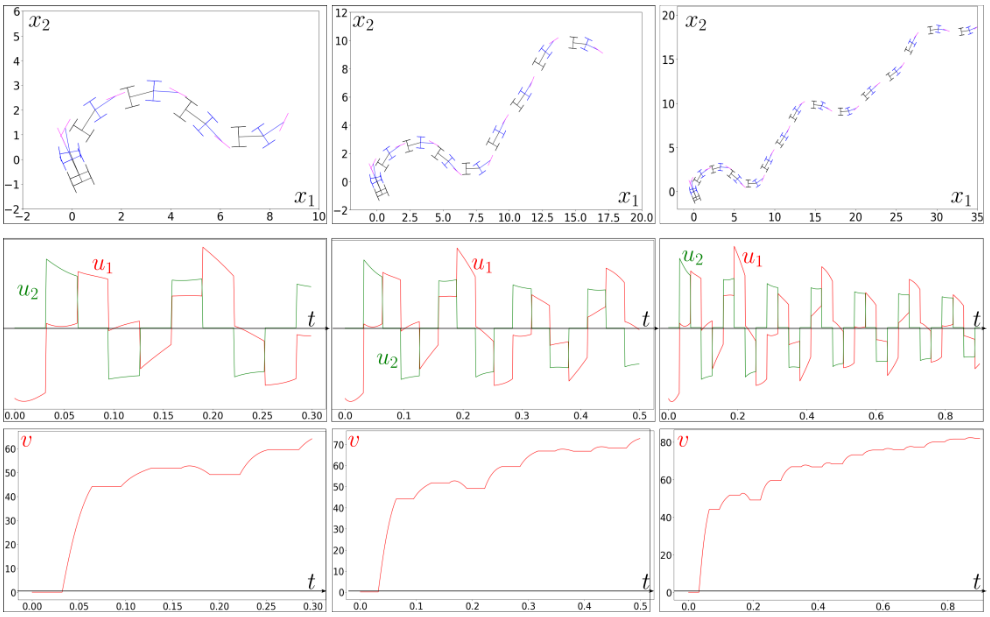
$$\begin{pmatrix} \dot{v} \\ \dot{\beta} \end{pmatrix} = K \begin{pmatrix} \bar{v} - v \\ \bar{\beta} - \beta \end{pmatrix}$$

where K is a high gain. Thus we take

$$\mathbf{a} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}}_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \left(K \begin{pmatrix} \bar{v} - v \\ \bar{\beta} - \beta \end{pmatrix} \right)$$

We take $\bar{\beta} = \text{sawooth}(\bar{\theta} - \theta)$ to have the right heading.

The simulation below illustrates the behavior of the controller for a desired heading $\bar{\theta} = \frac{\pi}{6}$ and a desired speed $\bar{v} = 100$.



The source code associated with the example are given below.

```
from roblib import *
def f(x,u):
   mx,my,θ,v,α,β=list(x[0:6,0])
   u1,u2=list(u[0:2,0])
   return array([[v*cos(θ)],[v*sin(θ)],[v*β],[-(β+sin(α))*u2],[-v*(β+sin(α))],[u1]])
def draw_snake(x):
   mx,my,θ,v,α,β=list(x[0:6,0])
   M0=array([[ 0  ,-0.3, 0.3, 0  ,0  ,-0.3,0.3,0  ,0,1],
            [-0.4,-0.4,-0.4,-0.4,0.4, 0.4,0.4,0.4,0,0]])
   M0=add1(M0)
   W=array([[-0.4,0.4],[0,0],[1,1]])
   R1=tran2H(mx,my)@rot2H(θ)
   M1=R1@M0
   skate=R1@tran2H(1,0)@rot2H(arctan(β))@W
   M2=R1@rot2H(α)@tran2H(-1,0)@M0
   plot2D(M1,'blue',1)
   plot2D(skate,'magenta',1)
   plot2D(M2,'black',1)
s=10
ax = init_figure(-s,s,-s,s)
x=array([[0],[0],[2],[0.5],[0],[0]]) #x,y,θ,v,α,β
dt=0.001
δ=sqrt(dt)
θbar=pi/6
βbar=0
vbar=100
for t in arange(0,1,dt):
     clear(ax)
     draw_snake(x)
     i=int(t/δ)
     def control_u(a):
        r=10
        a1,a2=list(a[0:2,0])
        eps= (1/r)*sign(a2)
        u1 = array([[eps],[0]])
        u2 = array([[0],[r]])
        u3 = array([[-eps],[0]])
        u4 = array([[0],[-r]])
        U=list([u1,u2,u3,u4])
        u=sqrt(4*abs(a2)/δ)*U[i%4]+array([[a1],[0]])
        return u

     mx,my,θ,v,α,β=list(x[0:6,0])
     βbar=0.95*βbar-0.03*sawtooth(θ-θbar)
     a=array([[0,1],[-1,0]])@(array([[5*(vbar-v)],[15*(βbar-β)]]))
     u=control_u(a)
     x=x+dt*f(x+(dt/2)*f(x,u),u)
```