Computing the viability kernel with pattern of cyclic trajectories

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Problematic

Consider the following nonlinear system:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

where $x \in \mathbb{R}^n$ states, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ control. $y \in \mathbb{R}^p$ is some output.

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We want to determine which states are viable with respect to some constraints on the output space.

Viability kernel

Viability Kernel

Given a system f,h and consider a sub-space of the output space $\mathcal{Y}$ and its inverse image in the state space $\mathcal{X} = h^{-1}(\mathcal{Y})$. The viability kernel $\mathrm{Viab}_{f,h}(\mathcal{Y}) \subset \mathcal{X}$ is a sub-space of the state space such that there exists a control u yielding $y = h(x)$ to stay within $\mathcal{Y}$ indefinitely.

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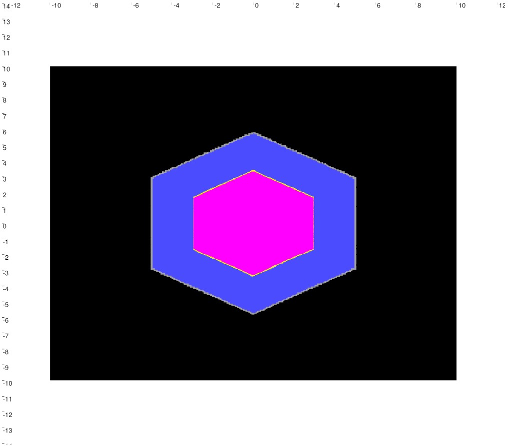
In robotics: y is the pose of the robot and $\mathcal{Y}$ is delimited by obstacles. Which states are safe ?

Illustration

- Forbidden
- Region frontier
- Not viable
- Unkown
- Viable



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Can be viewed as differential inclusion ($\mathcal{U}$ closed).
• Viability Kernel Algorithm [Saint-Pierre, Math. and Optim., 1994]: discretization of time and state spaces. Iterative approximation of the kernel (smh reachability analysis).

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• Viability Kernel Algorithm [Saint-Pierre, Math. and Optim., 1994]: discretization of time and state spaces. Iterative approximation of the kernel (smh reachability analysis).

• Robotics: determining *Inevitable Collision States* from set of evasive trajectories (complement to viability kernel). Typically, breaking trajectories or cyclic (circular) trajectories (see ref. in [Bouguerra et al., ICRA 2015]).
Adaptation of Saint-Pierre algorithm [Bouguerra et al., ICRA 2015].

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Hence, use predefined trajectories through patterns: functions that are zero on the corresponding trajectories.

- use of cyclic trajectories, and corresponding patterns, which implies viability;
- patterns allow to discard "time": the problem can be posed as a CSP.

Cyclic trajectories

Definition

Denote $\phi(x, t, u)$ a feasible trajectory starting from x and with control $u : \mathbb{R} \rightarrow \mathcal{U}$ ($\phi(x, 0, u) = x$). It is cyclic if $\exists \bar{t} > 0$ such that $\phi(x, \bar{t} + t', u) = \phi(x, t', u)$, and $u(\bar{t} + t') = u(t')$, for any $t' \geqslant 0$.

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Main property

If a state x admit a cyclic trajectory ϕ such that $\phi(x, t, u) \in \mathcal{X}, \forall t \leqslant \bar{t}$, then $x \in \mathrm{Viab}_{f,h}(\mathcal{Y})$.

Pattern

Let $V : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ be a convex function in the state $\times$ output space.

Definition

Given a control $u : \mathbb{R} \rightarrow \mathcal{U}$, V is a pattern for trajectory $\phi(x, t, u)$ if $V(x, h(\phi(x, t, u))) = 0, \forall t$.

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- for $p = 2$, if V is a pattern for a cyclic trajectory, w.l.g, $V \leqslant 0$ is the interior of the cycle
- for larger $p > 2$, several patterns can be used to describe a single trajectory

Pattern and viability

Property

Let V be a pattern function of some cyclic trajectory. Given states x, if $V(x, y) = 0 \implies y \in \mathcal{Y}$, then $x \in \mathrm{Viab}_{f,h}(\mathcal{X})$.

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$$\mathcal{V} := \{x : \forall y, V(x,y) \neq 0 \vee y \in \mathcal{Y}\} \subseteq \mathrm{Viab}_{f,h}(\mathcal{Y})$$

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Quantified constraint satisfaction

Example

Dubins car:

$$\left\{\begin{array}{rcl} \dot{x_1} & = & \cos(\theta) \\ \dot{x_2} & = & \sin(\theta) \\ \dot{\theta} & = & u \end{array}\right. ,$$

(position and heading), $u \in [-1, 1]$. We observe the position, i.e. $y_1 = x_1$ and $y_2 = x_2$.

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(position and heading), $u \in [-1, 1]$. We observe the position, i.e. $y_1 = x_1$ and $y_2 = x_2$.
We know that for $u \neq 0$ constant, the car follows a cyclic (circular) trajectory:

$$\begin{cases} x_1(t) = \frac{\sin(ut+\theta(0))}{u} - \frac{\sin(\theta(0))}{u} + x_1(0) \\ x_2(t) = -\frac{\cos(ut+\theta(0))}{u} + \frac{\cos(\theta(0))}{u} + x_2(0) \\ \theta(t) = ut + \theta(0) \end{cases}$$

(note: θ modulo 2π)

Example: pattern

We have the following pattern:

$$V(x_1, x_2, \theta, y_1, y_2) := \left(x_1 - \frac{\sin(\theta)}{u} - y_1\right)^2 + \left(x_2 + \frac{\cos(\theta)}{u} - y_2\right)^2 - \frac{1}{u^2} = 0.$$

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Note that $(x, \theta) \notin \mathcal{X}(\equiv \mathcal{Y}) \implies (x, \theta) \notin \mathrm{Viab}_{f,h}(\mathcal{Y})$. $P(x_1, x_2) \leqslant 0$ is necessary for $(x, \theta) \in \mathrm{Viab}_{f,h}(\mathcal{Y})$.

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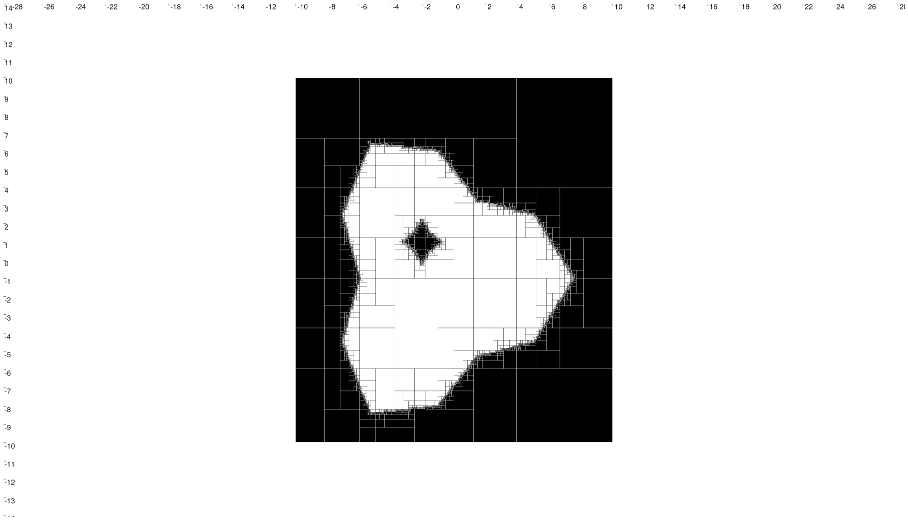
$\Rightarrow$ Build a paving of $\mathcal{Y}$. Inner boxes: initial conditions; boundary and outer boxes: domains of y.

Example: solving



Example: paving of $\mathcal{Y}$.

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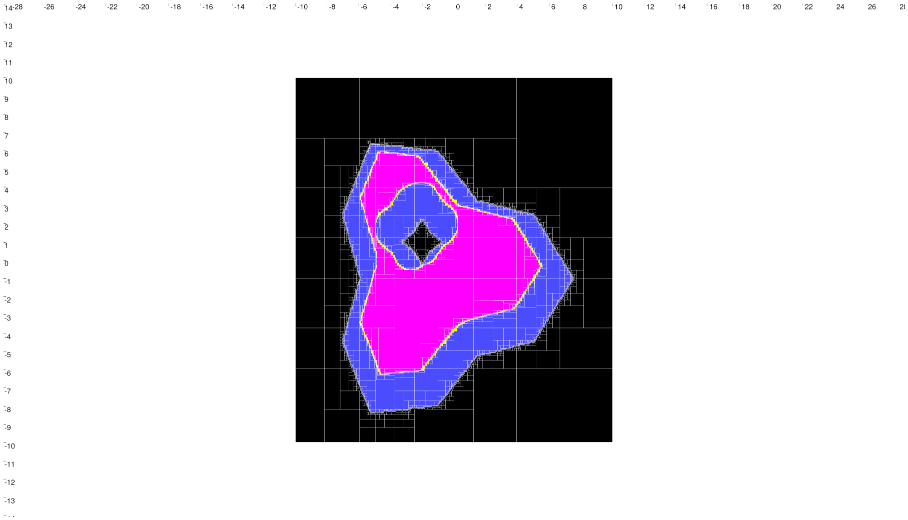
- $\forall\ \theta$ in its domain.

Other remarks:

- implementation in Ibex, using a separator on the quantified constraint,
- for convenience: consider $V \leqslant 0$ instead of $V = 0$,
- Cpu times on this instance: 155 s for the sampled version, 35 s on the original domain.

Example: computation



$$\theta \in [-\pi, -\frac{3}{4}\pi]$$

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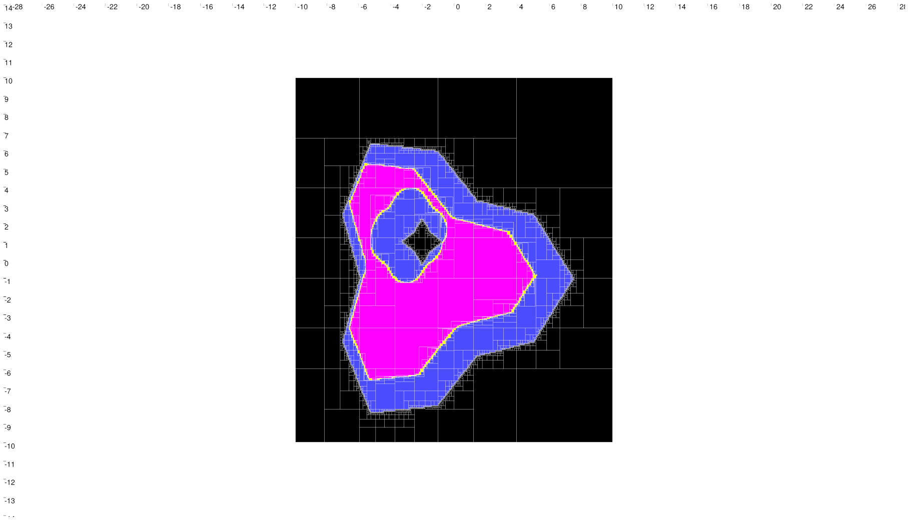

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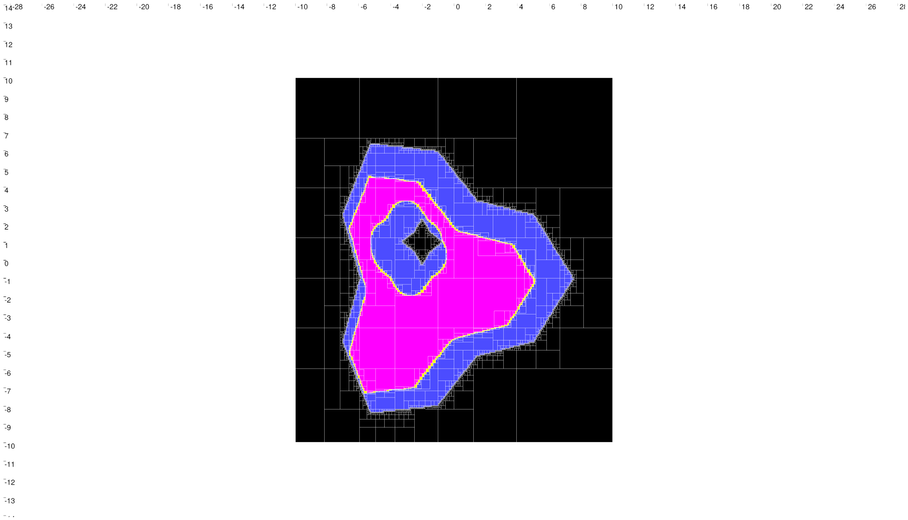

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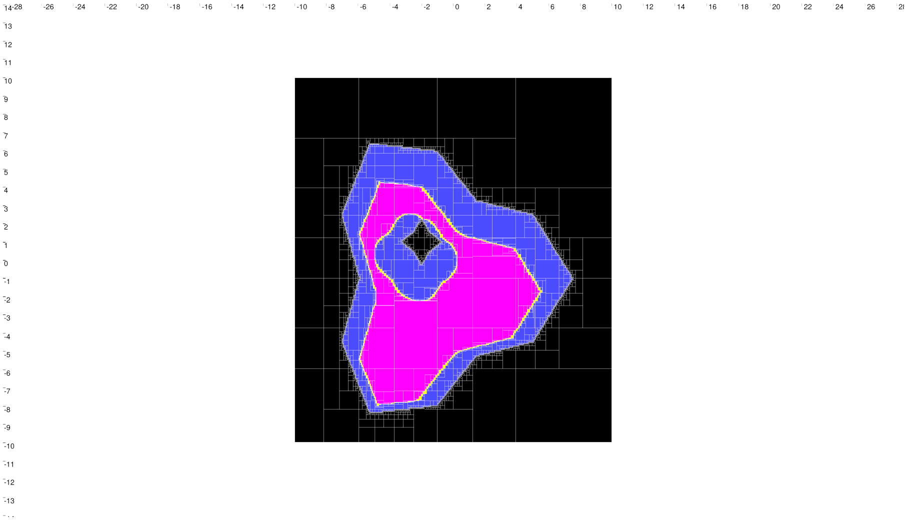

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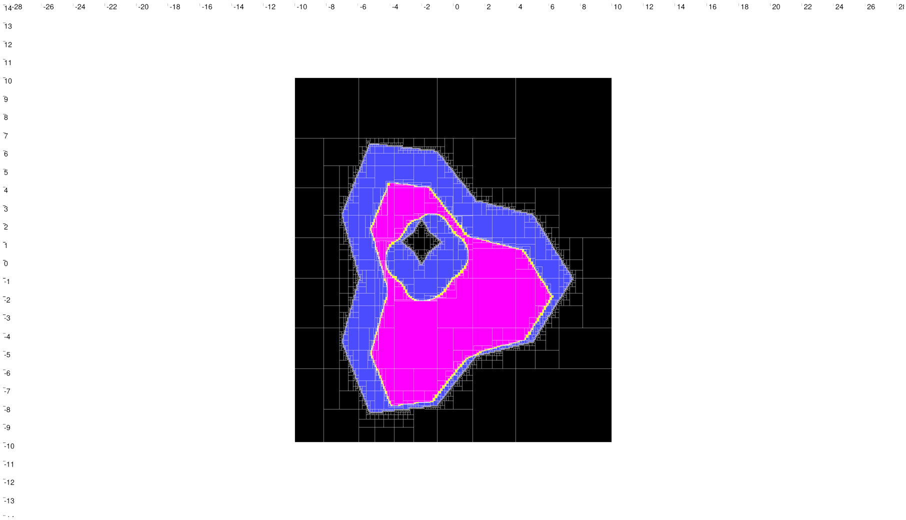

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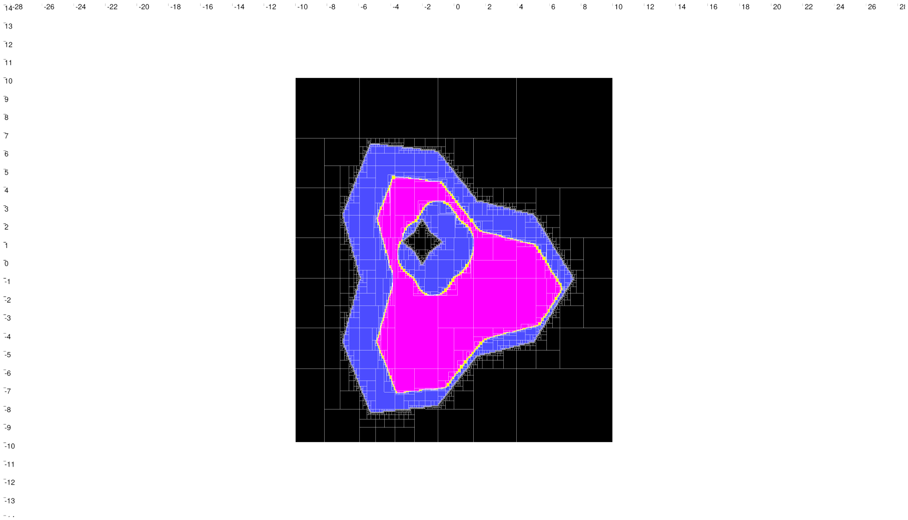

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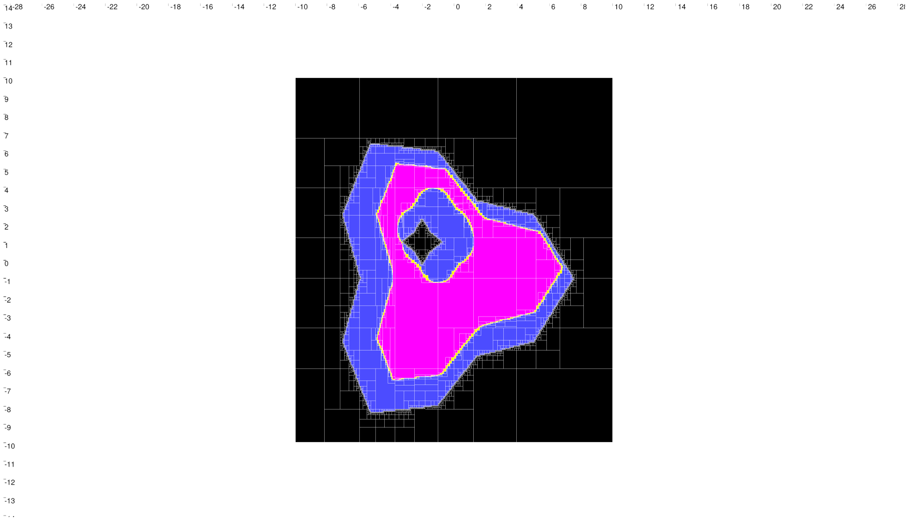

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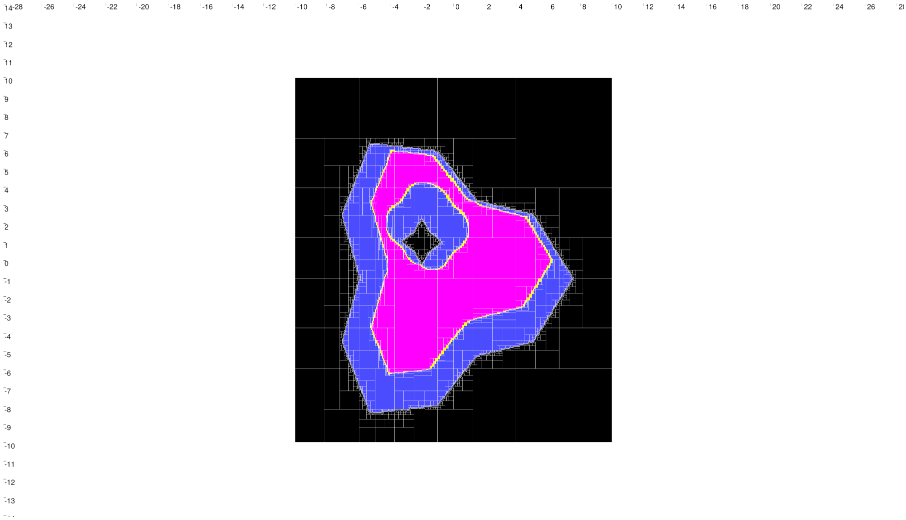

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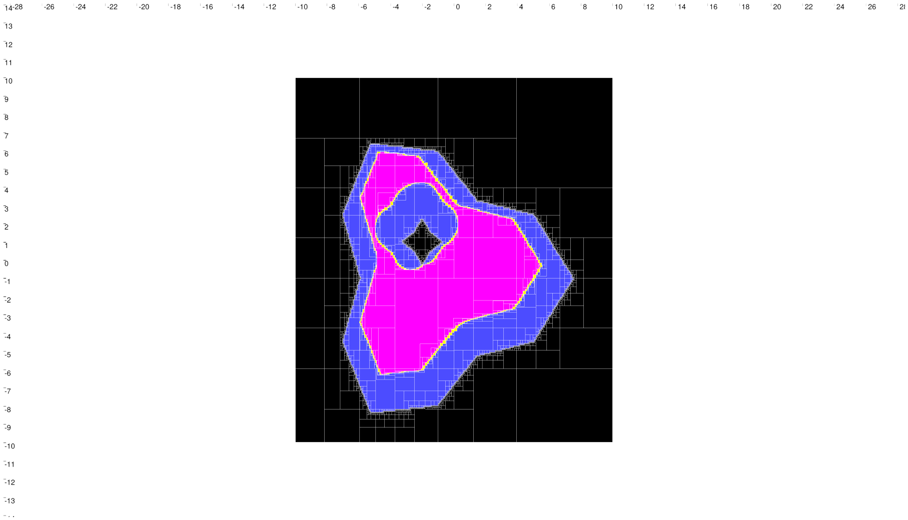

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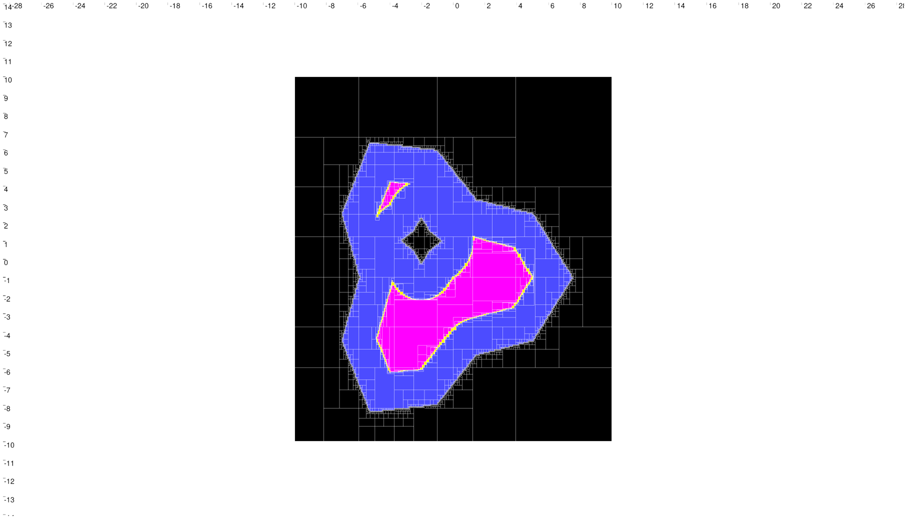

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- Generating V ?

Generating V ? An example

On the previous example:

$$V := (x_1 - c_1)^2 + (x_2 - c_2)^2 - d^2,$$

where c_1, c_2 and d are parameters depending on initial conditions of the states. In particular the flow satisfy $V = 0$.

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Hence,

$$\dot{V} = 2g(x_1 - c_1) + 2h(x_2 - c_2) = 0,$$

where $g = \cos(\theta), h = \sin(\theta)$.

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where $g = \cos(\theta), h = \sin(\theta)$.
Going further:

$$\ddot{V} = u\left(2g(x_2 - c_2) - 2h(x_1 - c_1)\right) + 2 = 0,$$

entailing $u = -2/\left(2g(x_2 - c_2) - 2h(x_1 - c_1)\right)$

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This control ensure the non variation of $\dot{V}$. We can note $\dot{u} = 0$ (due to $\dot{V} = 0$). It can be fixed to a $\tilde{u} \neq 0$ in $\mathcal{U}$.

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Hence, the following system (where x_1, x_2, θ must be viewed as initial conditions).

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Solution: $c_1 = x_1 - \frac{\sin(\theta)}{\tilde{u}}$, $c_2 = x_2 + \frac{\cos(\theta)}{\tilde{u}}$, $d = \frac{1}{\tilde{u}}$.

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Smarter ways, E.g. starting from given "time" cyclic trajectories ?

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- allows contraction of domains, i.e. captures accurately e.g. the boundary of the approximation,
- promising performances...
- ... but needs a comparison with other approaches (e.g. based on verified integration)