Station keeping problem

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Hybrid autonomous systems

Consider the system:

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$ states, $u \in \mathbb{R}^m$ control (discrete number of possible controls). System is autonomous as u only depends on states.

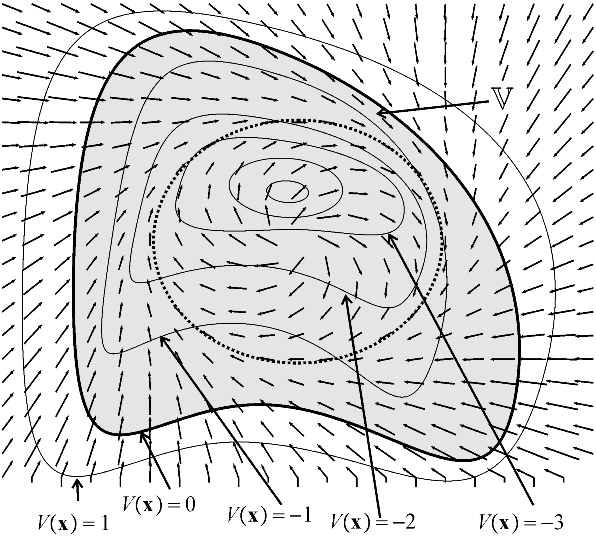
V-stability

A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is V-stable if:

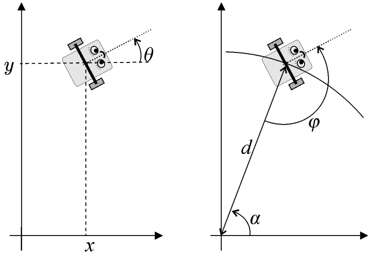
$$V \geqslant 0 \to \dot{V} < 0$$

Let $\mathbb{V} := \{x : V(x) \leqslant 0\}$. If the system is V-stable, then from any initial states, after a given time, the trajectory enters the set $\mathbb{V}$ and never exits it.

Illustration (Luc Jaulin)



Problem example: station keeping of a planar robot (1)



Cartesian coordinates

$$\begin{cases} \dot{x} = \cos(\theta) \\ \dot{y} = \sin(\theta) \\ \dot{\theta} = u \end{cases}$$

Polar coordinates

$$\begin{cases} \dot{\phi} = \frac{\sin\phi}{d} + u \\ \dot{d} = -\cos\phi \\ \dot{\alpha} = -\frac{\sin\phi}{d} \end{cases}$$

with $\phi - \theta + \alpha = \pi$.

Problem example: station keeping of a planar robot (2)

$$\begin{cases} \dot{\phi} & = & \frac{\sin\phi}{d} + u \\ \dot{d} & = & -\cos\phi \end{cases}$$

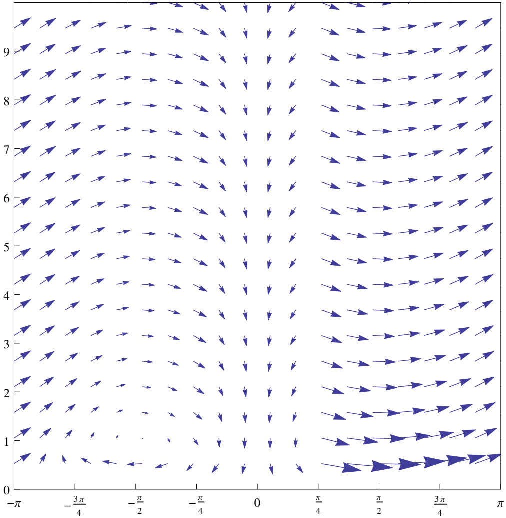
with control law, e.g:

$$u = \begin{cases} 1 & \text{if } \cos\phi \leqslant \frac{\sqrt{2}}{2} \\ -\sin\phi & \text{otherwise} \end{cases}$$

Question

Is it certain that from any initial state, the robot eventually stays around a beacon centred at the origin ?

Problem example: station keeping of a planar robot (3)



Transformation into a polynomial system

Idea

Try to prove the existence of an (algebraic) invariant for the system which allows to show what we want.

Only possible with algebraic systems, thus adding extra variables: $h = \sin\phi$, $g = \cos\phi$ and $e = \frac{1}{d}$. We obtain a polynomial system:

$$\begin{cases} \dot{h} &= (he+u)g \\ \dot{g} &= -(he+u)h \\ \dot{\phi} &= he+u \\ \dot{d} &= -g \\ \dot{e} &= ge^2 \end{cases}$$

with $h^2 + g^2 = 1$ and $de = 1$.

Darboux polynomials

Let $\mathcal{L}_f(p)$ be the Lie derivative of polynomial $p \in \mathbb{R}[x]$, (with respect to the flow f).

Definition

A polynomial p is Darboux if $\mathcal{L}_f(p) = qp$ with $q \in \mathbb{R}[x]$.

In our case, is Darboux:

- e with cofactor ge,
- $g^2 + h^2$ with cofactor 0.
- If u constant, $(u + 2eh)$ with cofactor $2ge$
- If $u = -h$, dh with cofactor $-g$.
- If $u = -h$, he with cofactor $g(2e - 1)$

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Darboux polynomials can be used to derive invariant/variant as rational or logarithmic functions, e.g. [Goubault et al., ACC 2014].

Invariant

Consider u constant, then since $u + 2eh$ and e^2 are Darboux with same cofactor, then

$$\mathcal{L}_f\left(\frac{u+2eh}{e^2}\right) = \frac{\mathcal{L}_f(u+2eh)e^2 - (u+2eh)\mathcal{L}_f(e^2)}{e^4} = 0.$$

This implies $\frac{u+2eh}{e^2}$ is constant (invariant). We have then for any initial condition e_0, h_0:

$$\frac{u+2eh}{e^2} = \frac{u+2e_0h_0}{e_0^2} \equiv -e_0^2(u+2eh) + e^2(u+2e_0h_0) = 0,$$

and as $e, e_0 > 0$ that the sign of $u + 2eh$ is maintained.

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→ Can be obtained e.g. with [Goubault et al., ACC 2014] or [Ghorbal and Platzer, TACAS 2014]

Invariant regions

$$\mathcal{V}_{cst} := \left\{ (\phi, d) : u + 2\frac{\sin(\phi)}{d} < 0 \right\}$$

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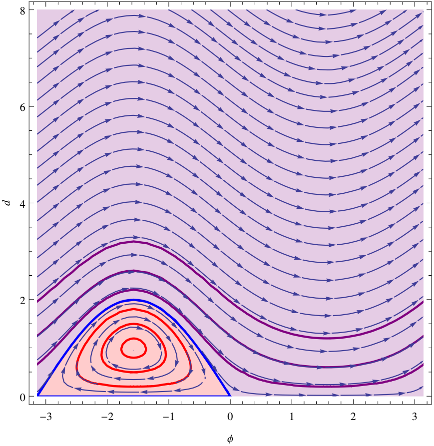

Invariant: proportional control

Recall: dh is Darboux with cofactor $-g$, and $\dot{d} = -g$. We can deduce:

$$\mathcal{L}_f \left(\log(|dh|) - d\right) = \frac{\mathcal{L}(|dh|)}{|dh|} + g = 0.$$

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$$\log(|dh|) - d = \log(|d_0 h_0|) - d_0 \equiv \log(|dh|) - d - \log(|d_0 h_0|) + d_0 = 0.$$

Invariant regions

$\mathcal{V}_{pro} := \{(\phi, d) : \log(|\sin(\phi)d|) - d < -2\}$

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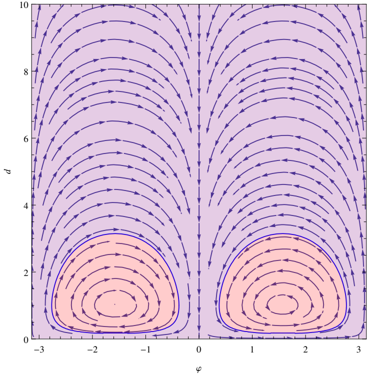

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Recall:

$$u = \begin{cases} 1 & \text{if } g \leqslant \frac{\sqrt{2}}{2} \\ -h & \text{otherwise} \end{cases} .$$

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Consequence: all the proportional invariant regions are cut (not existing), but an invariant region in the constant case exists, maximal when $V_{cst} = -1/2$ (i.e., the region tangent at the vertical line at $\phi = -\pi/4$, at point $(-\pi/4, \sqrt{2}/2)$).

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We observe when proportional control is applied:

- $\dot{V}_{cst} < 0$;
- on the frontier of the region $\phi = -\pi/4$ and $\phi = \pi/4$: flow enter when $d > 1$, flow exit when $d < 1$.

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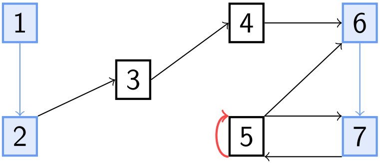
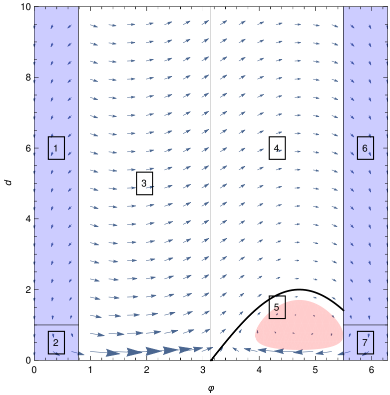
- $\dot{V}_{cst} < 0$;
- on the frontier of the region $\phi = -\pi/4$ and $\phi = \pi/4$: flow enter when $d > 1$, flow exit when $d < 1$.

$\Rightarrow$ Can the region defined by $V_{cst} \leqslant -1/2$ be reached from anywhere ?

Decomposition of the state space





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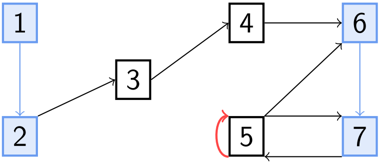

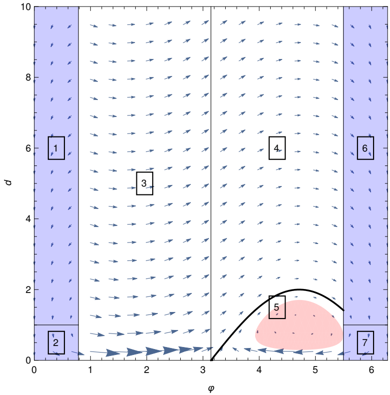

Regions 5,6,7 contains the invariant.

Analysis

From region 5 to 6: flow enters at $d_0 \in [1, \sqrt{2}]$. By construction, enters 7 and then 5 at d_1 satisfying:

$$\log(-d_0 h_0) - d_0 = \log(-d_1 h_0) - d_1 \equiv -d_1 e^{-d_1} = -d_0 e^{-d_0},$$

with $h_0 = \sin(-\pi/4)$.

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Lambert-W function $W(z)$ is defined as the solution to $z = xe^x$. Here, $d_1 = W(-d_0 e^{-d_0})$. From [Stewart, 2009], it satisfies:

$$2 - d_0 \leqslant -W(-d_0 e^{-d_0}) \leqslant 1/d_0.$$

Therefore, $d_1 \in [2 - d_0, 1/d_0] \to d_1 \in [2 - \sqrt{2}, 1]$.

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Recall $d_1 = \sqrt{(2)}/2$ belongs to the maximal invariant. Hence, two cases:

$$d_1 \in [2 - \sqrt{2}, \sqrt{2}/2] \quad \text{or} \quad d_1 \in [\sqrt{2}/2, 1]$$

First case: sliding mode

Sliding mode on surface $S = \{(\phi, d) : \phi = -\pi/4, d \in [\sqrt{2}/2, 1]\}$: flows in 5 are oriented towards 7 and vice versa.

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From [Fillipov, 1988; Liberzon, 2003], the system behaves as the unique convex combination $f_\lambda = \lambda f_- + (1-\lambda) f_+$ normal to the normal of the S (vector $(1,0)$). I.e. $f_{\lambda*}$ with λ^* satisfying $\langle f_{\lambda*}, (1,0) \rangle = 0$.

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$\Rightarrow f_{\lambda *} = (0, -g)$ with $-g = -\sqrt{2}/2$. d is strictly decreasing with constant speed: d eventually reaches $\sqrt{2}/2$

Second case

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Proof of V-stability of $V_{cst} + 1/2$

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- algebraic invariants can be automatically generated...
- ... but not the proof itself: only assisting the proof;
- in particular, specific functions (e.g. Lambert W function) can be involved;
- can the proof be more direct ?