Examen localisation, ENSTA-Bretagne, **ENSI 2**. 13 janvier 2015,

Polycopié et notes manuscrites autorisés. Calculatrice interdite. Durée: 1h15.

Exercise 1. We consider two random variables a, b of $\mathbb{R}$ that are independent and centered (i.e., $\bar{a} = \bar{b} = 0$). Their variance is $\Gamma_a = E((a-\bar{a})^2) = 1$ et $\Gamma_b = E((b-\bar{b})^2) = 4$. Define $c = 2a - 3b + 4$.

1) Give the expected value $\bar{c}$ and the variance Γ_c of c.

2) Define the random vector $\mathbf{x} = (a, b, c)^T$. Give the expected value $\bar{\mathbf{x}}$ and the covariance matrix $\mathbf{\Gamma_x}$ for $\mathbf{x}$.

Exercise 2. We consider the model

$$y(t) = p_1 t^2 + p_2 t + p_3 \cdot t^2 (t-1) + \beta(t)$$

where $\mathbf{p} = (p_1, p_2, p_3)$ is the parameter vector and β is a white noise, centered and with a variance $\Gamma_\beta = 4$. At different times t, we collect the measurements $y(t)$ as given by the following table:

t	-1	0	1	2	3
y	4	2	-1	3	5

1) Give an expression of the least-square estimation of the parameters p_1, p_2, p_3. Give the value of the matrices involved in your calculus.

2) We assume that the prior values for the parameters is $p_1 = p_2 = p_3 = 0$ with a covariance matrix equal to $10^4 \cdot I$ where I is the identity matrix. Using a linear unbiased orthogonal estimator, provide an expression of the estimation $\hat{\mathbf{p}}$ of $\mathbf{p}$ and provide the associated covariance matrix. If you prefer, you can also write directly the Matlab code.

3) Assume that you can use the Matlab function

```
[xhat1,Gx1]=kalman(xhat,Gx,u,y,Galpha,Gbeta,A,C);
```

as seen in the lesson. Write a small Matlab program which provides in one step an estimation of $\mathbf{p}$, with the associated covariance matrix.

4) Answer once more to Question 3, but now, in 5 steps (since we have 5 measurements).

Correction de l'examen du 13 janvier 2015

Correction of Exercise 1

1) We have $\bar{c} = 2\bar{a} - 3\bar{b} + 4 = 4$ and

$$\Gamma_c = E((c - \bar{c})^2) = E((2a - 3b)^2) = 4E(a^2) + 9E(b^2) + 12E(a \cdot b) = 4 + 36 = 40.$$

2) Since $E((a - \bar{a})(c - \bar{c})) = E(a(2a - 3b)) = 2$ and

$$E((b - \bar{b})(c - \bar{c})) = E(b(2a - 3b)) = -3E(b^2) = -12.$$

We have:

$$\bar{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \text{ and } \Gamma_{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & -12 \\ 2 & -12 & 40 \end{pmatrix}.$$

Correction of Exercise 2. 1) We have

$$\underbrace{\begin{pmatrix} 4 \\ 2 \\ -1 \\ 3 \\ 5 \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 2 & 4 \\ 9 & 3 & 18 \end{pmatrix}}_{\mathbf{C}} \mathbf{p} + \boldsymbol{\beta}.$$

For a least-square estimation, the MATLAB code is the following

```
y=[4;2;-1;3;5]; C=[1 -1 -2;0 0 0;1 1 0;4 2 4; 9 3 18 ]
phat=inv(C'*C)*C'*y;
```

2) The Matlab code is the following. $\texttt{G0} = 10000 * \texttt{eye}(3,3)$; $\texttt{p0} = [0;0;0]$;
$\texttt{Gbeta} = 4 * \texttt{eye}(5,5)$; $\texttt{S} = \texttt{C} * \texttt{G0} * \texttt{C}' + \texttt{Gbeta}$; $\texttt{K} = \texttt{G0} * \texttt{C}'/\texttt{S}$; $\texttt{ytilde} = \texttt{y} - \texttt{C} * \texttt{p0}$;
$\texttt{phat} = \texttt{p0} + \texttt{K} * \texttt{ytilde}$; $\texttt{Gp} = \texttt{G0} - \texttt{K} * \texttt{C} * \texttt{G0}$;
We get the same value for $\hat{\mathbf{p}}$ as for Question 1.

3) We get $\texttt{A} = \texttt{eye}(3,3); \texttt{Galpha} = 0 * \texttt{eye}(3,3); \texttt{u} = 0 * \texttt{p0}$;
$[\texttt{phat}, \texttt{Gp}] = \texttt{kalman}(\texttt{p0}, \texttt{G0}, \texttt{u}, \texttt{y}, \texttt{Galpha}, \texttt{Gbeta}, \texttt{A}, \texttt{C})$;

The covariance matrix corresponds to $\texttt{Gp}$. The results are the same as for Question 2.

4) We have the following code:

```
phat = [0;0;0]; A = eye(3,3); Galpha = 0 * eye(3,3);
u = 0 * p0; Gp = G0; Gbeta = 4;
for k = 1 : 5,
[phat, Gp] = kalman(phat, Gp, u, y(k), Galpha, Gbeta, A, C(k, :));
end
```

The results are the same as for Question 3.
