Stability analysis

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Abstract

Stability analysis of nonlinear systems is one of the biggest issue in control theory. A solution is to build a capture tube which ensure state equation integration. The $cross - out$ conditions guarantee the stability of the system inside the capture tube. This study aim to develop a method to build a capture tube for a nonlinear system avoiding wrapping effect and trying to have a quick computation.

Throw the example of two nonlinear systems, one of the 2nd order and one of the 3rd order, we illustrated a method to build a capture tube avoiding wrapping effect. Thanks to the $cross - out$ conditions, we can build a capture tube which ensure the state equation integration. We avoid the wrapping effect and the time of the computation is not too long.

However, we have a method for system of the 2nd and 3rd order, but not for higher orders. For systems of the 2nd order the capture tube is built with lines, for systems of the 3rd order the capture tube is built with triangle surfaces, for systems of the 4th order we cannot extend this method.

Contents

Introduction

Stability analysis of nonlinear systems is one of the biggest issue in control theory. There are some methods, as using Lyapunov functions to prove the stability of the system, but finding Lyapunov functions for nonlinear systems is not a trivial matter [COD-1]. Another solution can be to build a capture tube [JAU-1] that ensure state equation integration. The robot cannot turn off the capture tube, so we know all possible positions of the robot. Current methods use guaranteed integration to find such sets, but they are computationally expensive and tend to have wrapping effect for longer propagation times. This study aim to develop a method to build a capture tube for a nonlinear system avoiding wrapping effect and trying to have a quick computation.

After generalities about stability, this document treats the cases of systems of 2nd and 3rd order. We will offer a method to build a capture tube for nonlinear systems of the 2nd and 3rd orders.

Chapter 1

Generalities about stability

A nonlinear system can generally be describe a state equation of the form :

$$S_f : \dot{x}(t) = f(x(t), t) \tag{1.1}$$

A tube $\mathbb{G}$ is a function which associates to each $t \in \mathbb{R}$ a subset of $\mathbb{R}^m$.
A tube $\mathbb{G}$ is said to be a capture tube for S_f if we have the following implication :

$$x(t_a) \in \mathbb{G}(\mathrm{t_a}), \tau > 0 \Rightarrow x(t_a + \tau) \in \mathbb{G}(\mathrm{t_a} + \tau) \tag{1.2}$$

Consider the tube

$$\mathbb{G}(.) : t \to \{x \mid \mathbf{g}(x, t) \leq 0\} \tag{1.3}$$

where $\mathbf{g}$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is assumed to be differentiable with respect to both $\mathbf{x}$ and t. The following theorem shows that the problem of proving that $\mathbb{G}$(t) is a capture tube can be cast to proving that a set of inequalities has no solution.

Theorem :If the system of constraints (called the $cross - out$ conditions)

$$\begin{cases} (i) & \dot{g}_i(x, t) \geq 0, \\ (ii) & g_i(x, t) = 0, \\ (iii) & \mathbf{g}(x, t) \leq 0, \end{cases} \tag{1.4}$$

is inconsistent (*i.e.*for all $\mathbf{x}$, all $t \geq 0$ and all $i \in$\{1,...,m\}, the inequalities are not satisfied), then $\mathbb{G}(.) : t \to \{x | \mathbf{g}(x, t) \leq \mathbf{0}\}$ is a capture tube for the system $\dot{x} = \mathbf{f}(x, t)$

Chapter 2

System of the 2nd order

2.1 Generalities

To build a capture tube for a system of the 2nd order, we use the example of the pendulum with the following state equations:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\sin(x_1) - 0.15 * x_2 \end{cases} \tag{2.1}$$

where x_1 is the angle of the pendulum and x_2 the rotational speed.

2.2 Construction of the initial tube

First, we have two initial vectors : $x_1 = [x_{1l} \ ; \ x_{1u}]$ and $x_2 = [x_{2l} \ ; \ x_{2u}]$. With these two vectors, we have four extrem points.

$x_A = (x_{1l} \ ; \ x_{2u})$; $x_B = (x_{1l} \ ; \ x_{2l})$; $x_C = (x_{1u} \ ; \ x_{2l})$; $x_D = (x_{1u} \ ; \ x_{2u})$

Then, we simulate the trajectories of these four initial points with Euler method, as shown on the figure 2.1.

In the case of the pendulum, we know the system will converge to a stable position, so the tube will be closed. With the Euler method, the trajectories of the extrem points are composed of lines.

We begin the construction of the tube from one of the extrem points and we put it in a list which will contain the points of the tube.

We want the best initial tube (*i.e.*, an initial tube containing all extrem trajectories) to save time for the treatment of the $cross-out$ conditions next. To chose the next point of the initial tube, we select the following point of the last point of the initial tube (*i.e.*, if the last point comes from the extrem trajectory A, the last point of the tube is $x_A(j)$ and the point selected is $x_A(j+1)$).

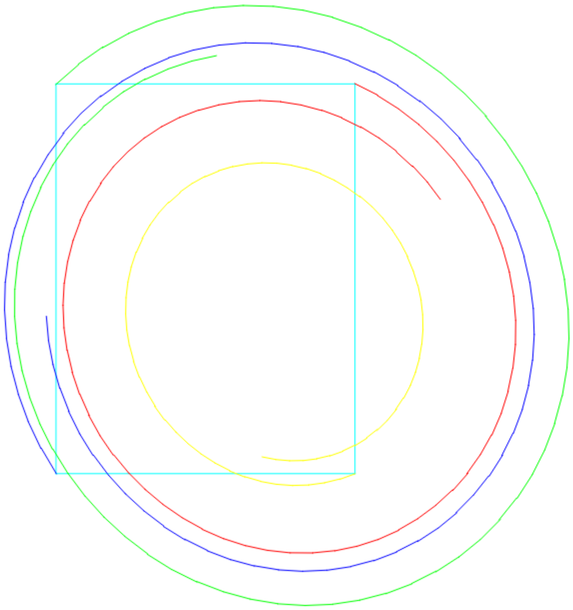

Figure 2.1: Extrem trajectories : initial box (cyan) and extrem trajectories (red, blue, green, yellow)

Then we check if there is a trajectory (trajectory of extrem points and side of the initial box) crossing the line $[x_A(j); x_A(j+1)]$. If it is the case, we add the last point of the line crossing the line $[x_A(j); x_A(j+1)]$ ($i.e.$, the two last points of the tube are $x_A(j)\, and\, x_B(k)$ for example)

We apply this method up to close the tube ($i.e.$, when we have twice the same point in the initial tube), the result is on the figure 2.2. The trajectory of the tube has to begin and finish with the same point, so the points of the tube added before the closed point are deleted because they are inside the tube. The result of the initial tube is shown on the figure 2.3.

2.3 Cross-out conditions

We will explain the $cross - out$ conditions for the pendulum system. In our case, the condition (iii) of the $cross - out$ conditions is guaranteed by the condition (ii) because we have only one tube. So, we only have two conditions to check.

The condition (ii) checks if the point selected is on the border or not. So, to satisfy this condition, we will take points on the border of the tube.

The condition (i) checks, thanks to the derived vector, if the point on the border of the tube will stay inside the tube or if it will go out of the tube. If the point will go out of the tube, the tube is not a capture tube.

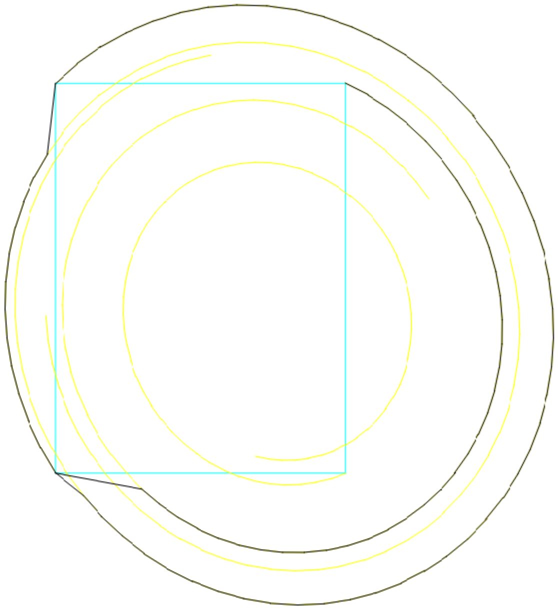

Figure 2.2: Closed tube : initial box (cyan), extrem trajectories (yellow) and initial closed tube (black)

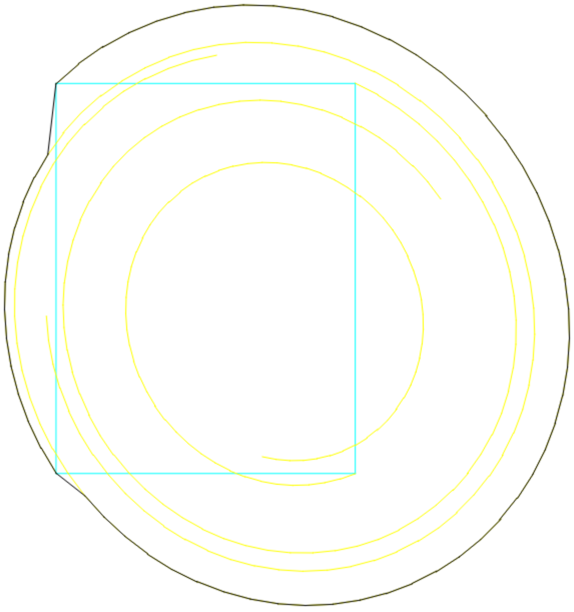

Figure 2.3: Initial tube : initial box (cyan), extrem trajectories (yellow) and initial tube (black)

2.4 Impementation of the method

The tube built is composed of lines, which form the border of the tube. So, we check the $cross - out$ conditions on each line of the tube.

To begin, we have two points (the extremities of the line). On this line we build a number of boxes, (if we build lots of boxes, the accuracy will be better but the computation time will be longer) in our case we chose to build ten boxes. Thanks to this, we are sure that we will check all points of the border.

First, we determine the equation of the line, we will use the slope of the line to check if the points inside the box will stay inside the tube or not.

Then, for each box of the line, we determine $\dot{x_1}$and $\dot{x_2}$thanks to interval method [JAU-2].

Now, we calculate the value of the angle between the border (with the slope of the line) and $\dot{X} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix}$. We just need the sign of this angle, thanks to this sign we know if the points inside the box will stay inside the tube or if they will go out of the tube.

For all boxes on the line, if all points stay inside the tube, we do nothing and we check the next line.

But if some points go out of the tube, we have to extend the tube. So, we move the point which correspond to the final point of the line. Then, we check the new line, if it is good, we keep this new line, but if it is not good, we move the point again, and we check the new line etc...

When we have to move the point, we move perpendicular to the initial line, as explained on the figure 2.4.

After checking all lines of the tube, we come back at the first point. But, maybe we moved the last point ($i.e.$, the first point) of the tube, so the first line moved. So, we will check again the tube, if the last point of the tube moved checking the cross-out conditions.

2.5 Results

The result of this method is satisfactory. The figure shows the result of this method with initial vectors $x_1 = [-0.3; 0.1]$ and $x_2 = [-0.2; 0.3]$ and $dt = 0.1$ for Euler method.

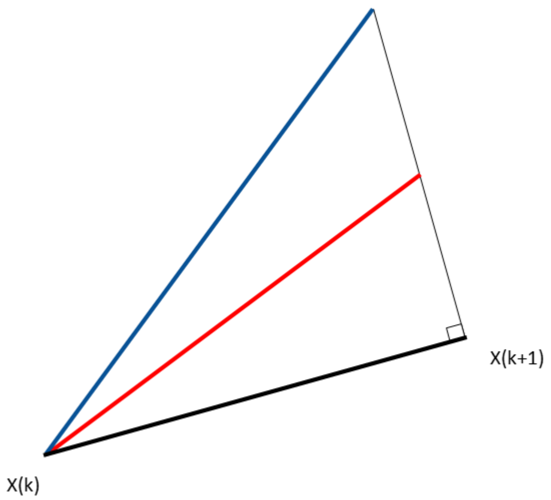

Figure 2.4: Moving point : the initial line (black line), after one move (red line), after a second move (blue line)

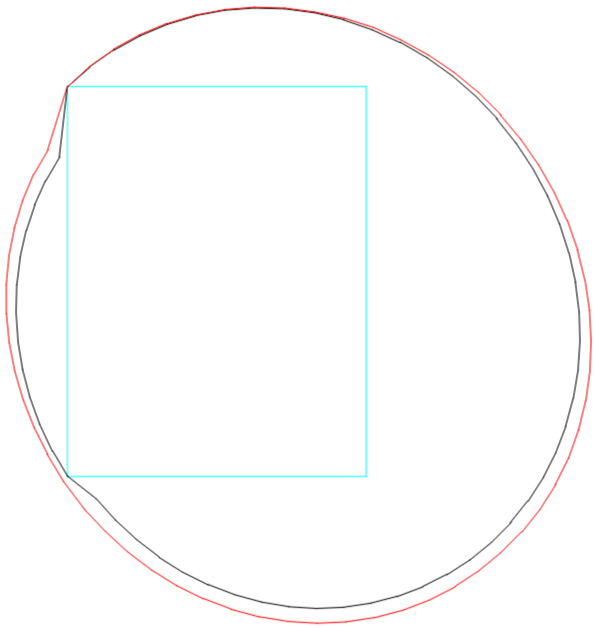

Figure 2.5: Capture tube pendulum system : initial box (cyan), initial tube (black), capture tube (red)

Chapter 3

System of the 3rd order

3.1 Generalities

To build a capture tube for a system of the 3rd order, we use the example of a nonlinear system with the following state equations:

$$\begin{cases} \dot{x_1} = -x_2 * x_3 + 1 \\ \dot{x_2} = x_1 * x_3 - x_2 \\ \dot{x_3} = x_3^2 * (1 - x_3) \end{cases} \tag{3.1}$$

3.2 Construction of the initial tube

First, we have three initial vectors : $x_1 = [x_{1l} \; ; \; x_{1u}]$, $x_2 = [x_{2l} \; ; \; x_{2u}]$ and $x_3 = [x_{3l} \; ; \; x_{3u}]$. With these three vectors, we have eight extrem points.

$x_A = (x_{1l} \; ; \; x_{2l} \; ; \; x_{3l})$; $x_B = (x_{1l} \; ; \; x_{2l} \; ; \; x_{3u})$; $x_C = (x_{1l} \; ; \; x_{2u} \; ; \; x_{3l})$; $x_D = (x_{1l} \; ; \; x_{2u} \; ; \; x_{3u})$; $x_E = (x_{1u} \; ; \; x_{2l} \; ; \; x_{3l})$; $x_F = (x_{1u} \; ; \; x_{2l} \; ; \; x_{3u})$; $x_G = (x_{1u} \; ; \; x_{2u} \; ; \; x_{3l})$ and $x_H = (x_{1u} \; ; \; x_{2u} \; ; \; x_{3u})$

Then, we simulate the trajectories of these eight extrem points with Euler method, as shown on the figure 3.1.

To begin, we watch the trajectories of these points and we choose the sides of the initial tube. To do this, we select the extrem points of the initial tube and we link them with lines. The tube has to be closed, and an example of this step is shown on the figure 3.2.

Then, to have a better accuracy, we take several points on the sides of the initial tube, thanks to this we will avoid wrapping effect.

Then, we calculate the positions of all points of the initial tube thanks to Euler method and we build surfaces between the points of the tube. We use triangles surfaces because we cannot always build rectangular surface between four points in 3D. So, we have two triangles to represent the surface between four points, as shown on the picture 3.3.

Now, we have the initial tube, that is a tube built thanks to triangles surfaces. On the figure 3.4 we can see a part of an initial tube.

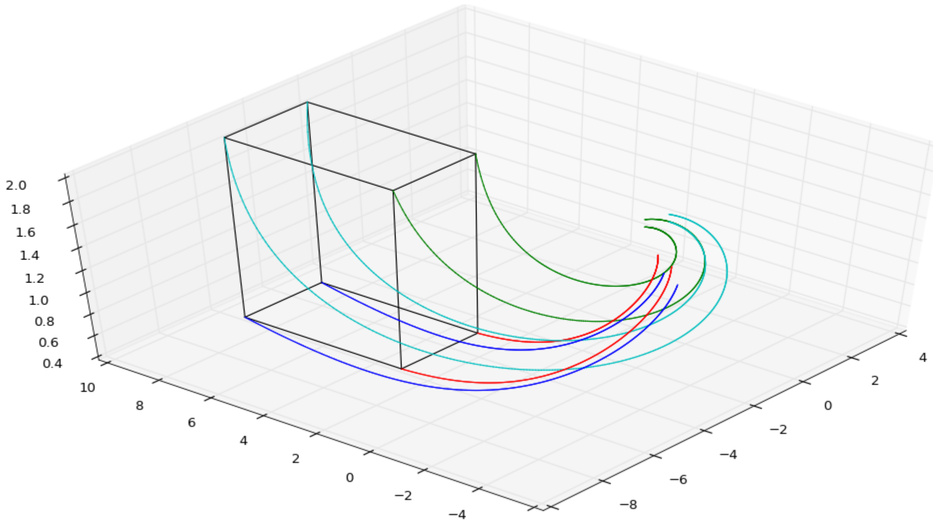

Figure 3.1: Extrem trajectories : initial box (black) and extrem trajectories (blue, red, cyan, green)

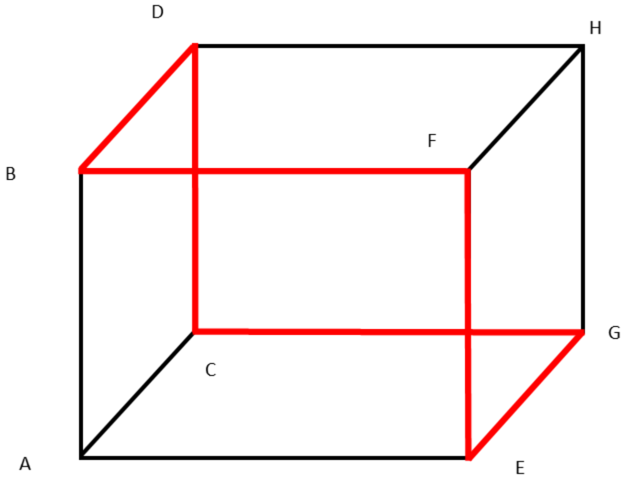

Figure 3.2: Sides of the initial tube (red), corners of the initial tube : E-F-B-D-C-G-E

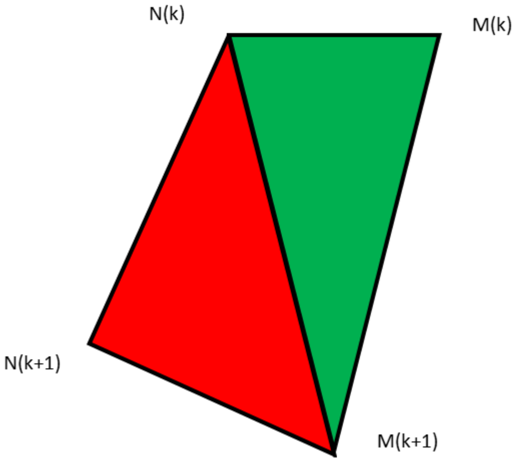

Figure 3.3: Triangles surfaces between four points

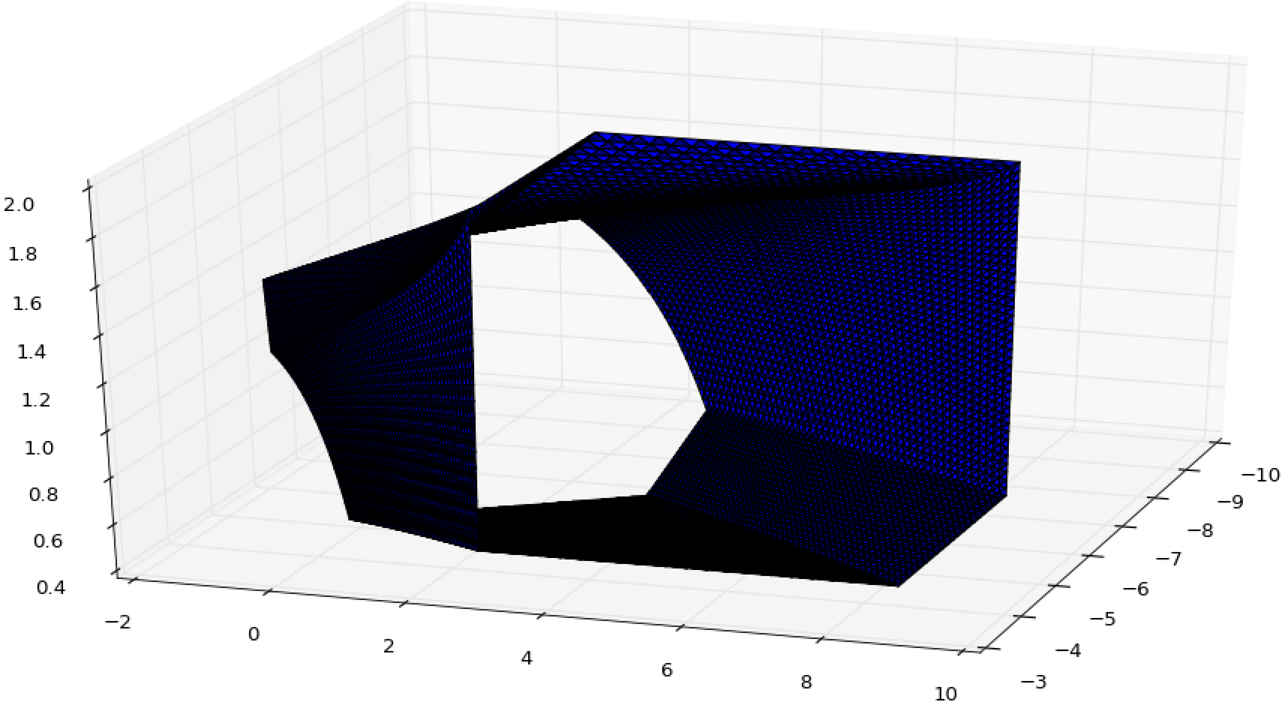

Figure 3.4: Part of an initial tube

3.3 Cross-out conditions

We will explain the $cross - out$ conditions for our system of the 3rd order. In our case, the condition (iii) of the $cross - out$ conditions is guaranteed by the condition (ii) because we have only one tube. So, we only have two conditions to check.

The condition (ii) checks if the point selected is on the border or not. So, to satisfy this condition, we will take points on the border of the tube.

The condition (i) checks, thanks to the derived vector, if the point on the border of the tube will stay inside the tube or if it will go out of the tube. If the point will go out of the tube, the tube is not a capture tube.

3.4 Implementation of the method

The tube built is composed of triangles surfaces, which form the border of the tube. So, we check the $cross - out$ conditions on each triangle surface of the tube.

To begin, we have three points (the extremities of the triangle surface). On this triangle, we build boxes, thanks to this we are sure that we check all points of the surface.

We build the boxes with an algorithm based on the SIIVA algorithm [JAU-2]. We start with a cube which includes the triangle. Then we check if the surface crosses the cube, if it is the case we cut the box in eight boxes and we check again up to have a box with the wanted size. Finally we have some boxes, with the same size, which approximate the triangle surface.

First, we calculate the equation of the surface. We will use the normal vector of the surface to check if the points inside the boxes will stay inside the tube or not.

Then, for each box of the surface, we calculate $\dot{x_1}$, $\dot{x_2}\, and\, \dot{x_3}$ thanks to the interval method [JAU-2].

Now, we calculate the value of the angle between the normal vector and the vector $\dot{X} = \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix}$. We just need the sign of the sine of this angle, thanks to this sign we can know if the points will stay inside the tube or if they will go out of the tube.

For all boxes, if all points stay inside the tube, we do nothing and we check the next triangle.

If some points go out of the tube, we have to extend the tube. So, we move one point of the triangle. Then, we check the new triangle, if it is good, we keep this new surface, but if it is not good, we move the point again and we check the new surface, etc...

When we have to move a point, we have two solutions if we are on a side of the initial tube or at a corner of the initial tube.

With a surface on a side of the initial tube, we move the point on the perpendicular direction, following the normal vector of the surface.

With a surface at a corner of the initial tube, if we use the same method we have wrapping effect so, we use another method and we take a vector which is the mean of the normal vectors of the two surfaces of the two sides of the corner as explained on the figure 3.5.

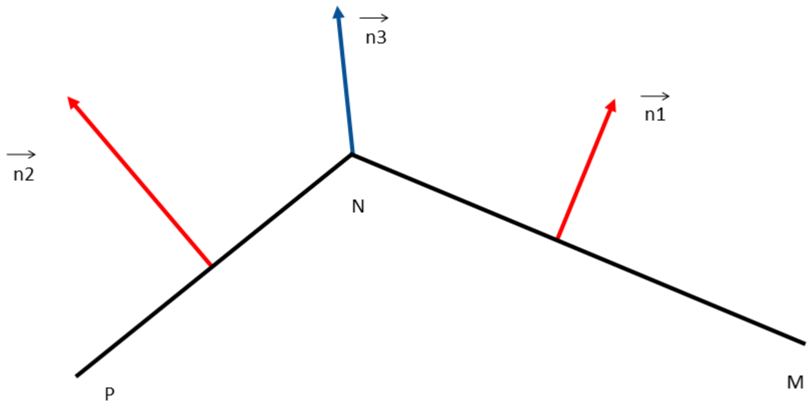

Figure 3.5: Move a point of a surface : $\vec{n_3}$ is the mean of $\vec{n_1}$ and $\vec{n_2}$, we move the point N following this direction

When we check the *cross − out* conditions of the tube, we begin at a point (for example the point M_1) and we check the following surface of the same step and, thanks to this, we check all the surfaces of the same step. This is explained on the picture 3.6.

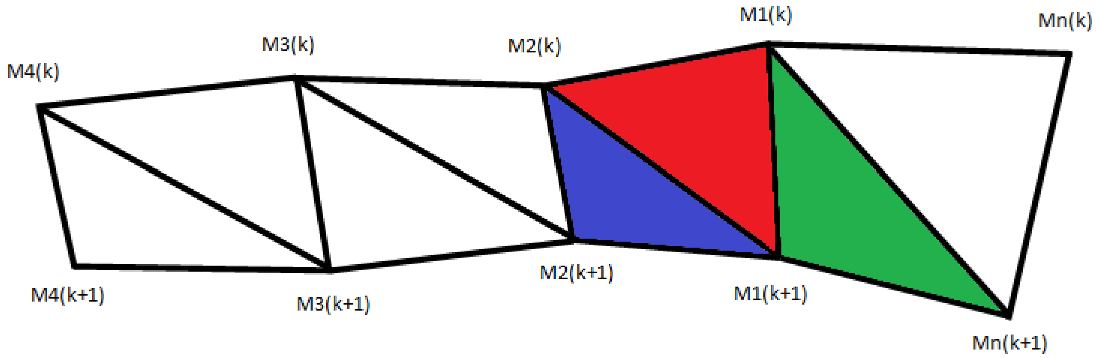

Figure 3.6: Checking a step. We begin with the red surface, we check the *cross − out* conditions on this surface and if we have to move a point, we move the point M_1(k+1). Then, we check the blue surface and if we have to move a point, we move the point M_2(k+1). We continue with the same method and finally we will check the green surface and move the point M_1(k+1) if it is necessary.

We start at the point M_1 and we finish at the point M_1 and if the point M_1 moved, the initial surface moved too. So we will check again the surfaces of the tube and if some points are moved, we check again up to moved no points. Then, we calculate the new values of the points of the next step with Euler method, because some points of the current step moved and then we can check the following step with the same method.

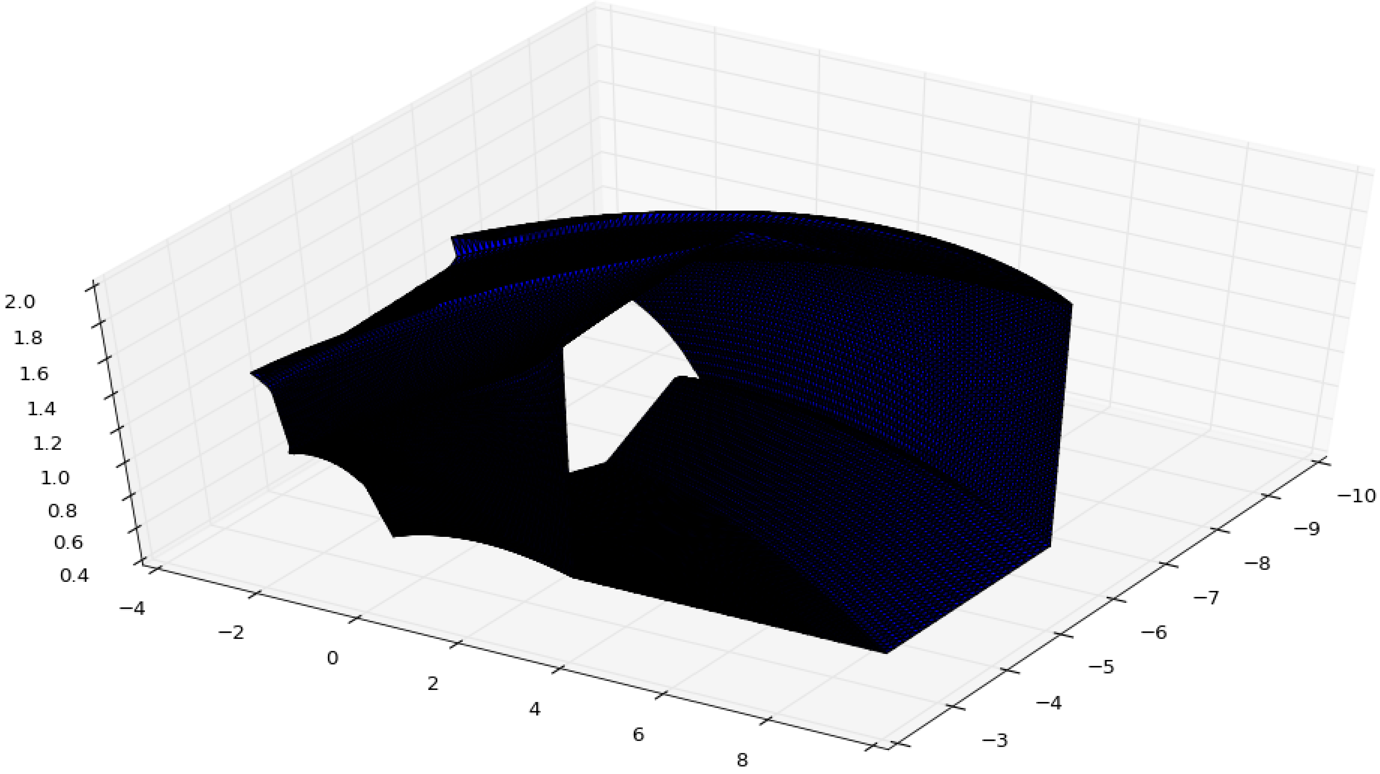

Figure 3.7: Part of a capture tube of a system of the 3rd order

3.5 Results

The result of this method is satisfactory. We have a disadvantage, we do not have a method to choose the initial sides of the tube, we have to look the simulation of the corners of the initial box and then, we choose the sides of the tube.

The figure 3.7 shows the result of this method with initial vectors $x_1 = [-6\,; -3]$, $x_2 = [3\,; 9]$ and $x_3 = [0.6\ ;\ 2]$. The simulation is 100 step with dt=0.01 and the time of the simulation is about 12 minutes.

The surface with triangles is the capture tube achieved by the method explained here.

Conclusion

Throw the example of two nonlinear systems, one of the 2nd order and one of the 3rd order, we illustrated a method to build a capture tube and avoid wrapping effect. Thanks to the $cross - out$ conditions, we can build a capture tube which ensure the state equation integration. We avoid the wrapping effect and the time of the computation is not too long.

However, we have a method for system of the 2nd and 3rd order, but not for higher orders. For systems of the 2nd order the capture tube is built with lines, for systems of the 3rd order the capture tube is built with triangle surfaces, for systems of the 4th order we cannot extend this method.

A futur work can be to find a method for system of the n-th order. After some researches, zonotopes or interpolation method seem to be difficult to implement. So, other matematical concepts have to be studied to implement the method using the $cross - out$ for system of the n-th order.

Bibliography

[COD-1] Codres E., Stability analysis using energy functions with perturbations, Second year report, University of Manchester, 2016

[JAU-1] Jaulin L., Lopez D., Le Doze V., Le Menec S., Ninin J., Chabert G., Saad Ibnseddik M., Stancu A., Computing capture tube, SCAN 2014

[JAU-2] Jaulin L., Kieffer M., Didrit O., Walter E., *Applied Interval Analysis*, Springer-Verlag, 2001

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RAPPORT D'EVALUATION
ASSESSMENT REPORT

Merci de retourner ce rapport dès la fin du stage à / *Please return this report at the end of the internship to :*
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I - ORGANISME / *HOST ORGANISATION*

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Fonction LECTURER

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Nom du stagiaire accueilli / *Name of the trainee* FLORIAN TANGUY

II - EVALUATION / *ASSESSMENT*

Veuillez attribuer une cote, en encerclant le chiffre approprié, pour chacune des caractéristiques suivantes. Cette note devra se situer entre **A (très bien)** et **F (très faible)**
Please give a mark between A (very good) and F (very weak).

MISSION / *TASK*

- La mission de départ a-t-elle été remplie ?
  Has the task been carried out well ? A [✓ A] B C D E F

- Le stagiaire a-t-il apporté des connaissances nouvelles à l'organisme d'accueil ?
  Did the trainee bring news skills to the host organisation ? ☐ oui/*yes* ☑ non/*no*

  Lesquelles ? *Which ones ?*

- Manquait-il au stagiaire des connaissances ? ☐ oui/yes ☑ non/no
  Was the trainee lacking skills ?

  Si oui, lesquelles ? / *Which ones ?*

ESPRIT D'EQUIPE / *TEAM SPIRIT*

- Le stagiaire s'est-il bien intégré dans l'organisme d'accueil / Did the trainee easily integrate into the host organisation?
  (disponible, sérieux, adaptation au travail de groupe)
  (flexible, dedicated, adapts himself (herself) to the team work) A [✓ A] B C D E F

Avez vous des observations ou suggestions à nous faire part / *Do you have any remarks or suggestions to comment ?* VERY INTELLIGENT AND HARDWORKING STUDENT. IT WAS A PLEASURE TO HAVE HIM IN MY RESEARCH GROUP.

COMPORTEMENT AU TRAVAIL / *BEHAVIOUR TOWARDS WORKS*

Le comportement du stagiaire était-il conforme à vos attentes *(Ponctuel, ordonné, respectueux, soucieux de participer et d'acquérir de nouvelles connaissances)* ?
Did the trainee come up to expectations (*Punctual, methodical , responsive to management instructions, concerned with quality, concerned about gaining new skills) ?* A [✓ A] B C D E F

Avez vous des observations ou suggestions à nous faire part / Have you any remarks or suggestions to comment ? NO ADITIONAL COMMENTS.

INITIATIVE – AUTONOMIE / *INITIATIVE – AUTONOMY*

Le stagiaire s'adaptait vite à de nouvelles situations ? A [✓ A] B C D E F
(Proposer des solutions aux problèmes rencontrés, autonome dans son travail,..)

Did the trainee adapt himself (herself) well to new situations ? A B C D E F
(Suggest solutions to problems met, was independent in his/her job…)

Avez vous des observations ou suggestions à nous faire part / Have you any remarks or suggestions to comment ? NO ADITIONAL COMMENTS

CULTUREL – COMMUNICATION / *CULTURAL – COMMUNICATION*

Le stagiaire était-il ouvert, d'une manière générale, à la communication A [✓ A] B C D E F
Was the trainee open to listening and expressing himself (herself)

Avez vous des observations ou suggestions à nous faire part / Have you any remarks or suggestions to comment ? NO ADITIONAL COMMENTS

OPINION GLOBALE / *OVERALL ASSESSMENT*

- La valeur technique du stagiaire était : A [✓ A] B C D E F
  The technical skills of the trainee were :

III - PARTENARIAT FUTUR / *FUTURE PARTNERSHIP*

- Etes-vous prêt à accueillir un autre stagiaire l'an prochain ?
  Are you prepared to host another trainee next year ? ☑ oui/*yes* ☐ non/*no*

Fait à 30 SEPTEMBER 2016 ,le ______ In
MANCHESTER ,on ______

Signature

Merci pour votre coopération
We thank you very much for your cooperation