Boundary approach to characterize the inner and outer approximation of the image of a disk

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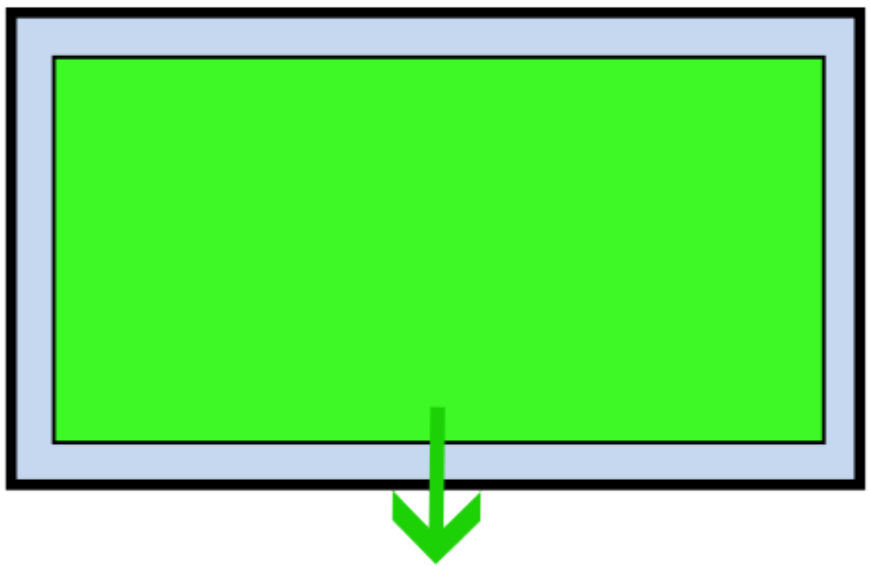
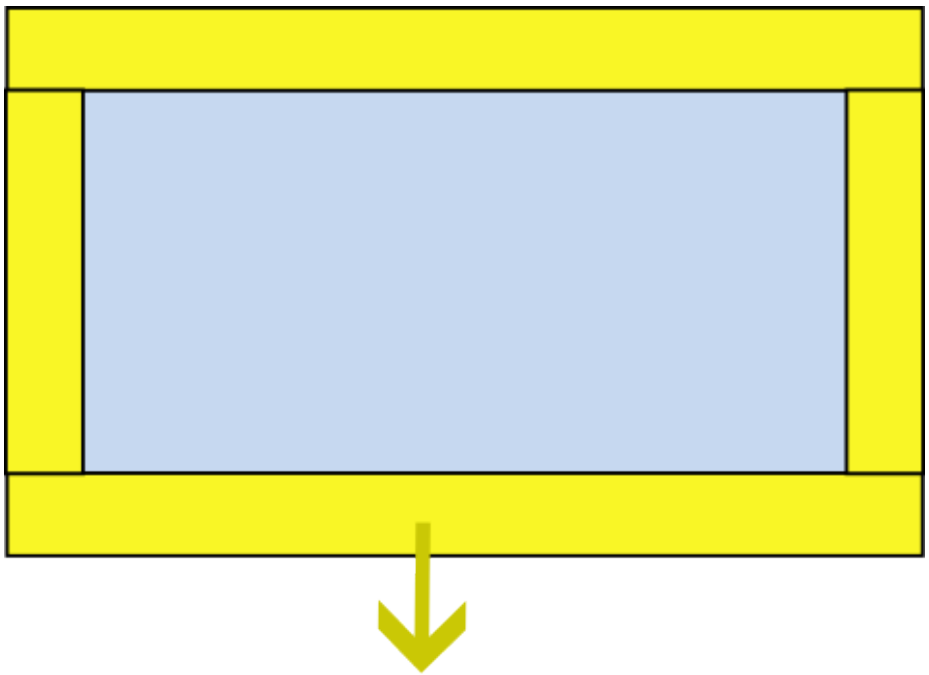
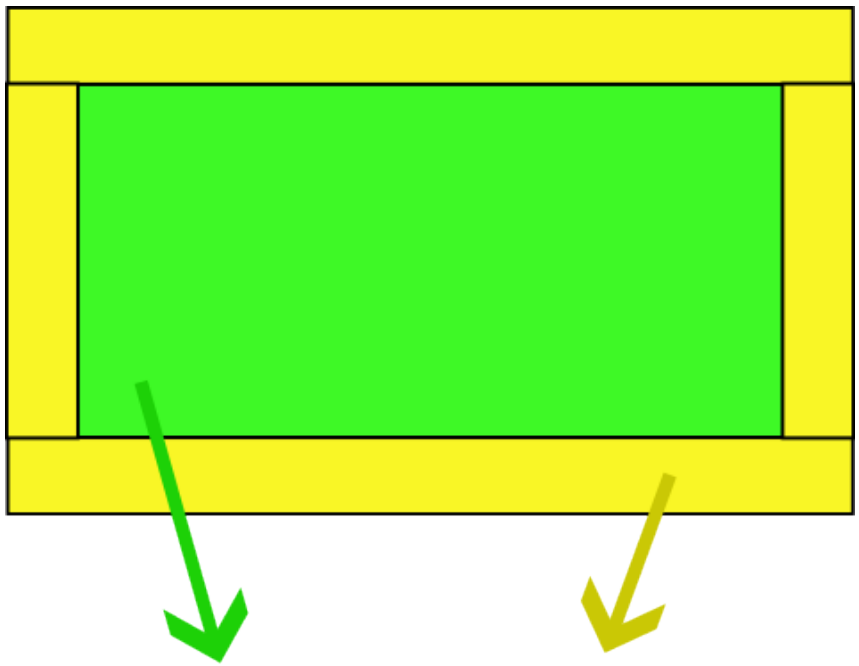
Inner and outer approximation of a set



Inner Approximation
of the rectangle



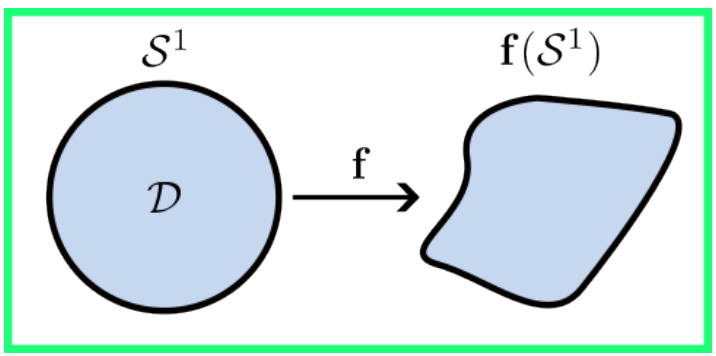
Outer Approximation
of the Boundary



Outer Approximation
of the rectangle

Notations and hypothesis

- $\mathcal{D}$ is the unit disk, its boundary is $\mathcal{S}^1$ the unit circle
- The function $\mathbf{f} : \mathcal{D} \to \mathbb{R}^2$ is C^1
- $\mathbf{f}$ has no singularity on $\mathcal{D}$



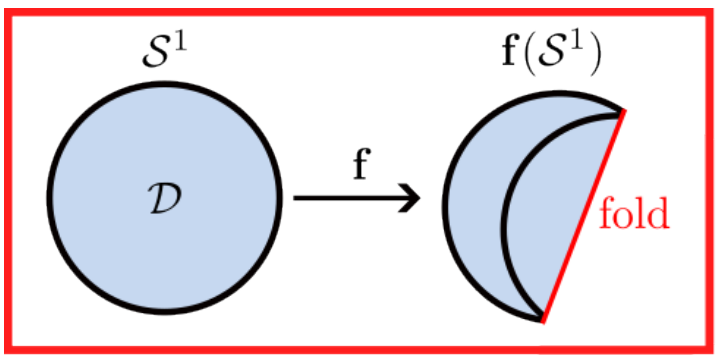
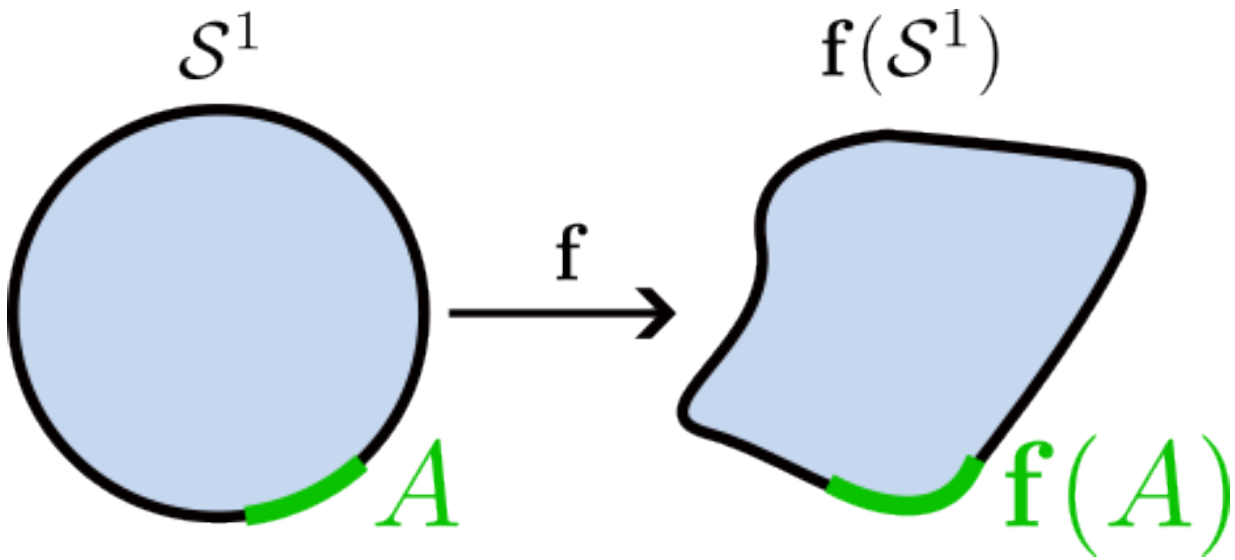
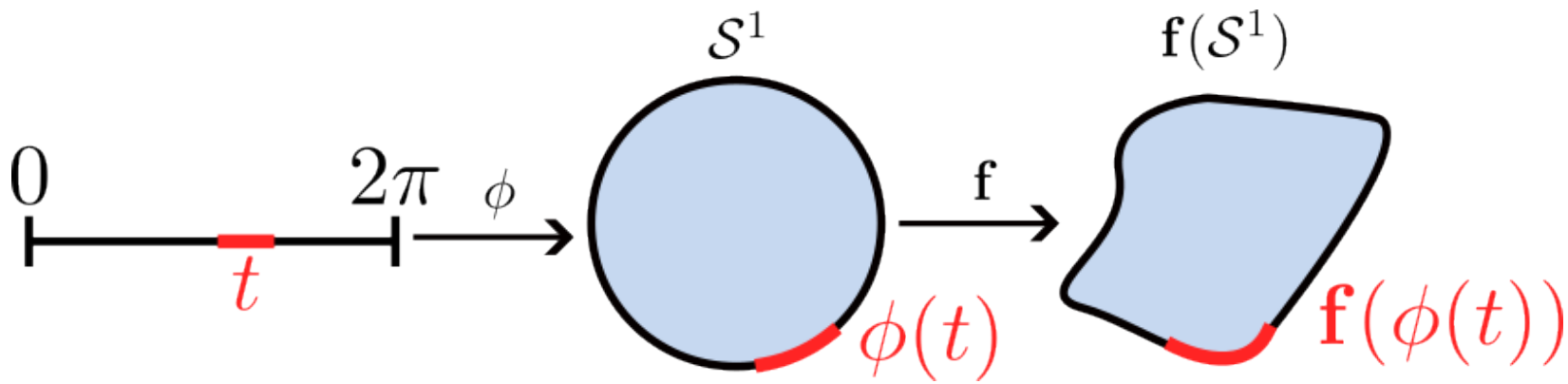


Image of the unit circle



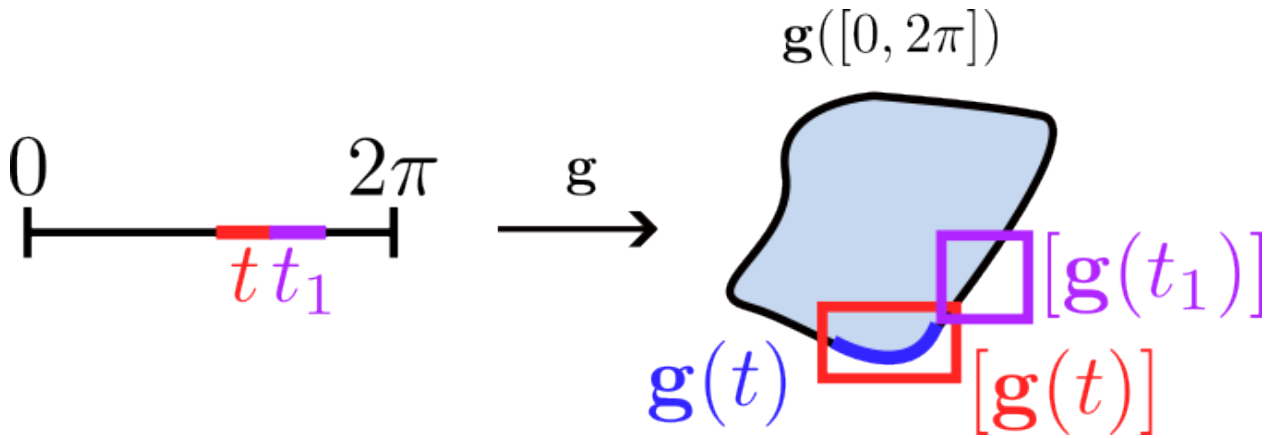
Boundary image

$$\begin{array}{rccc} \phi: & [0,2\pi] & \rightarrow & \mathcal{S}^1 \\ & t & \mapsto & \begin{pmatrix} cos(t) \\ sin(t) \end{pmatrix} \end{array}$$



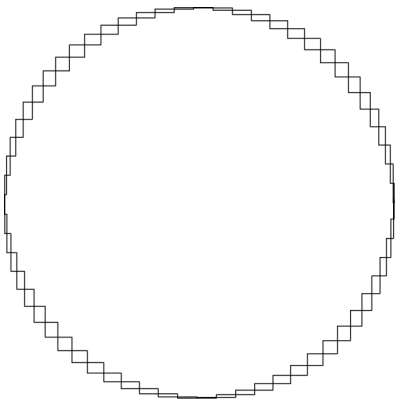
Function g

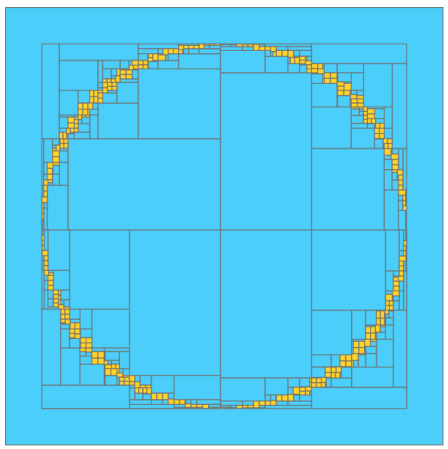
$$\begin{array}{rccc} g: & [0,2\pi] & \to & \mathbb{R}^2 \\ & t & \mapsto & \mathbf{f}\circ\phi(t) \end{array}$$

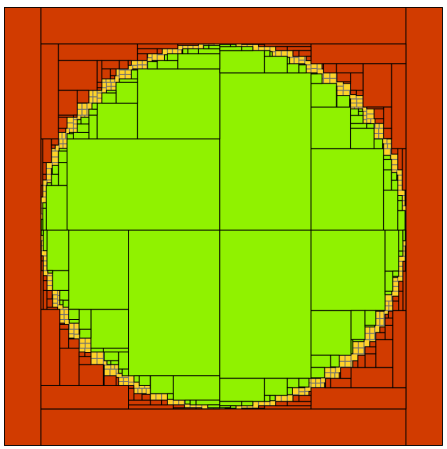


Simple case

f is the identity

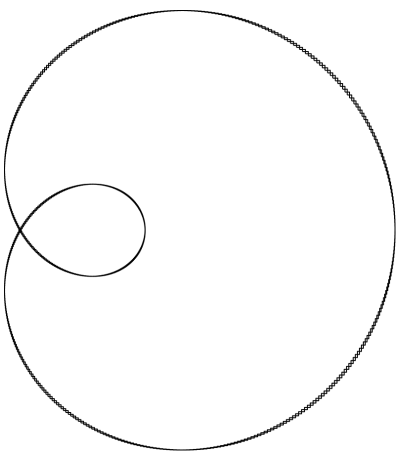


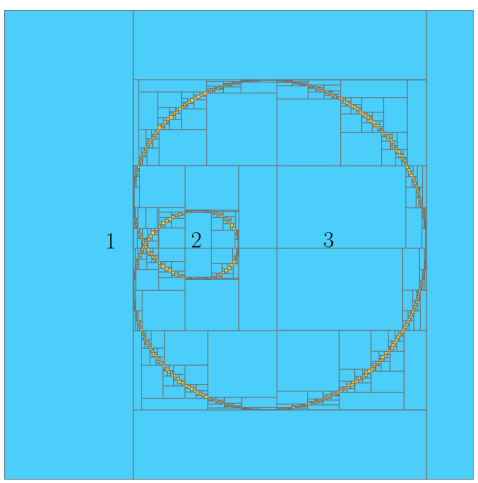


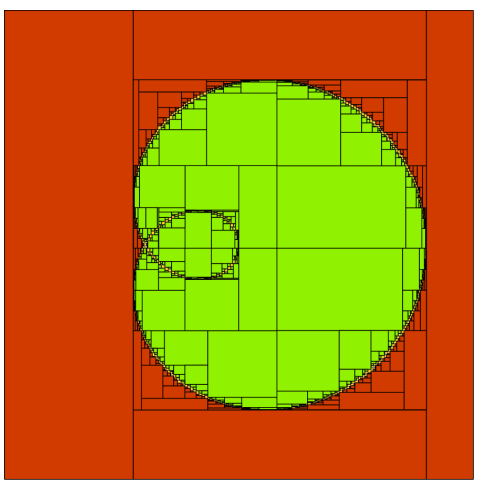


More complex case

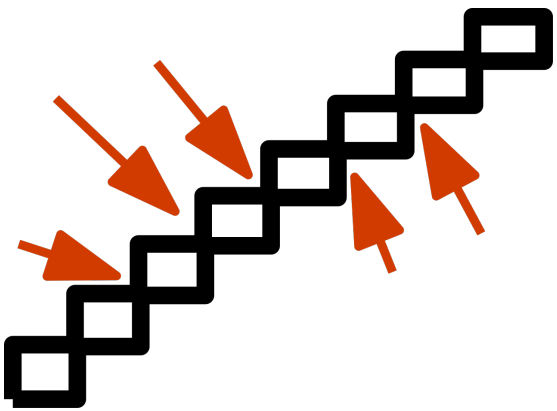
$$\forall \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^2 - x_2^2 + x_1 \\ 2x_1x_2 + x_2 \end{pmatrix}$$

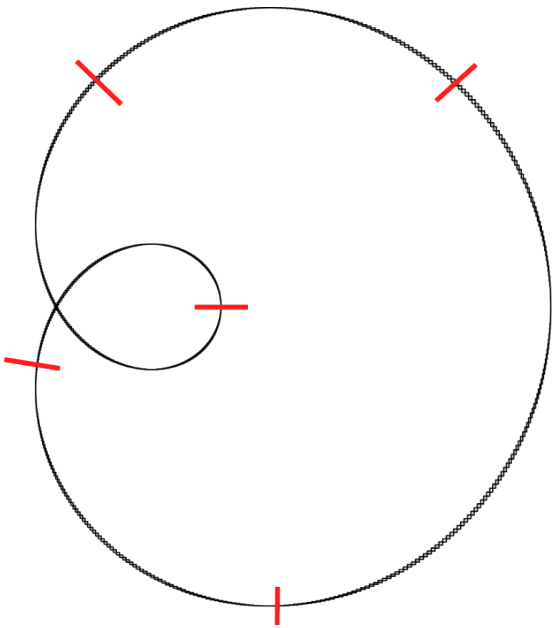




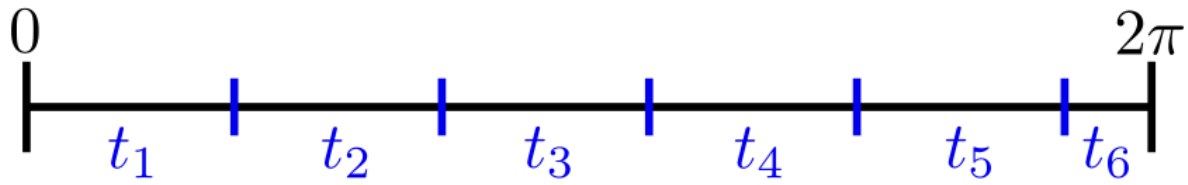


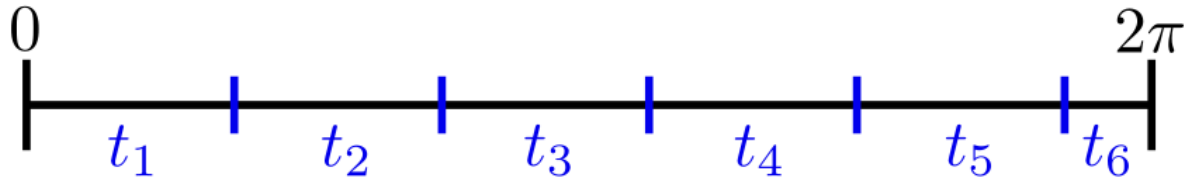
Problematic : finding intersections



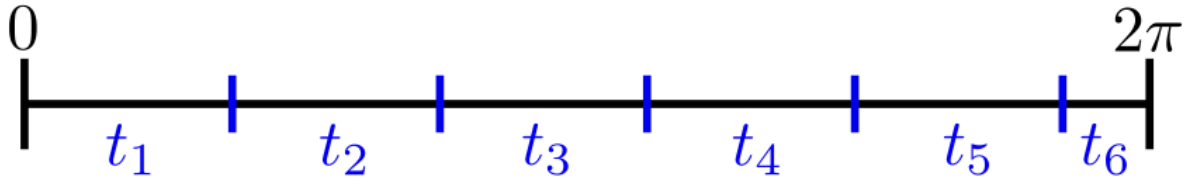


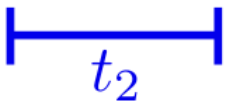
Box Chain decomposition



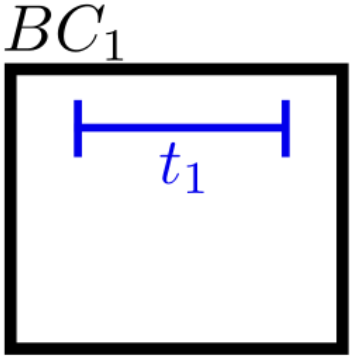

0
2π
t_1
t_2
t_3
t_4
t_5
t_6

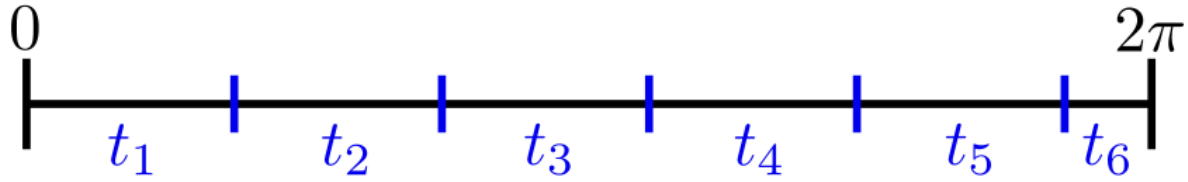

t_1
g injective over t_1?

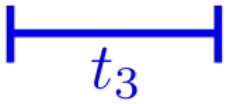




$\mathbf{g}$ injective over $t_1 \cup t_2$?
t_2 interesects t_1 ?

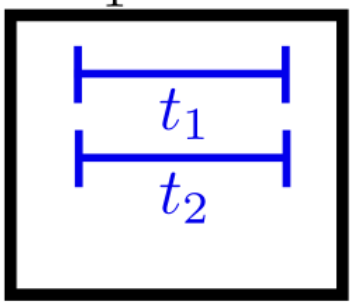


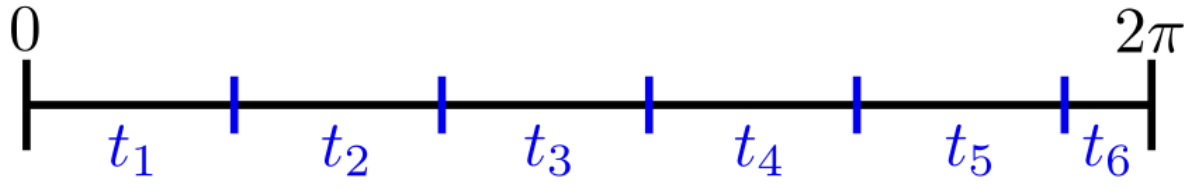


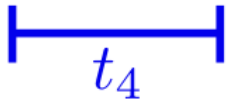


$\mathbf{g}$ injective over $t_1 \cup t_2 \cup t_3$?
t_3 interesects $t_1 \cup t_2$?

BC_1

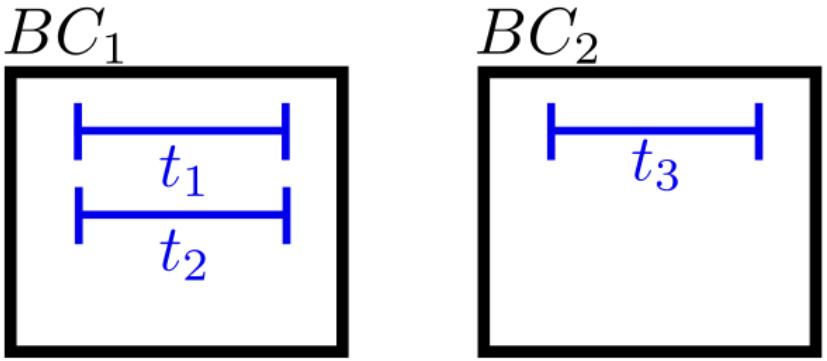


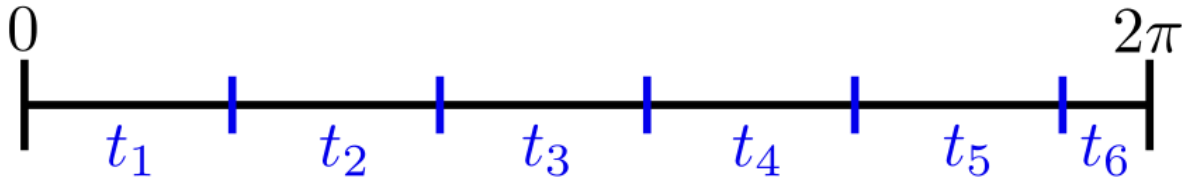


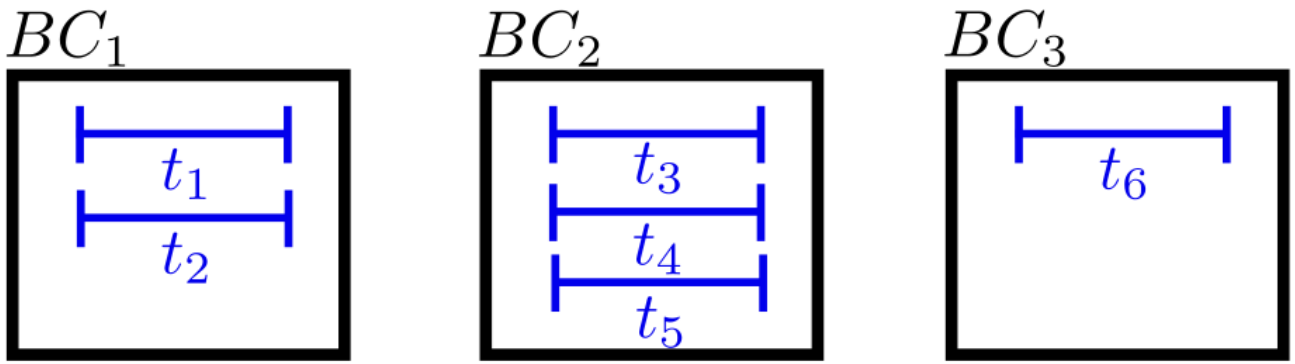


g injective over $t_3 \cup t_4$?

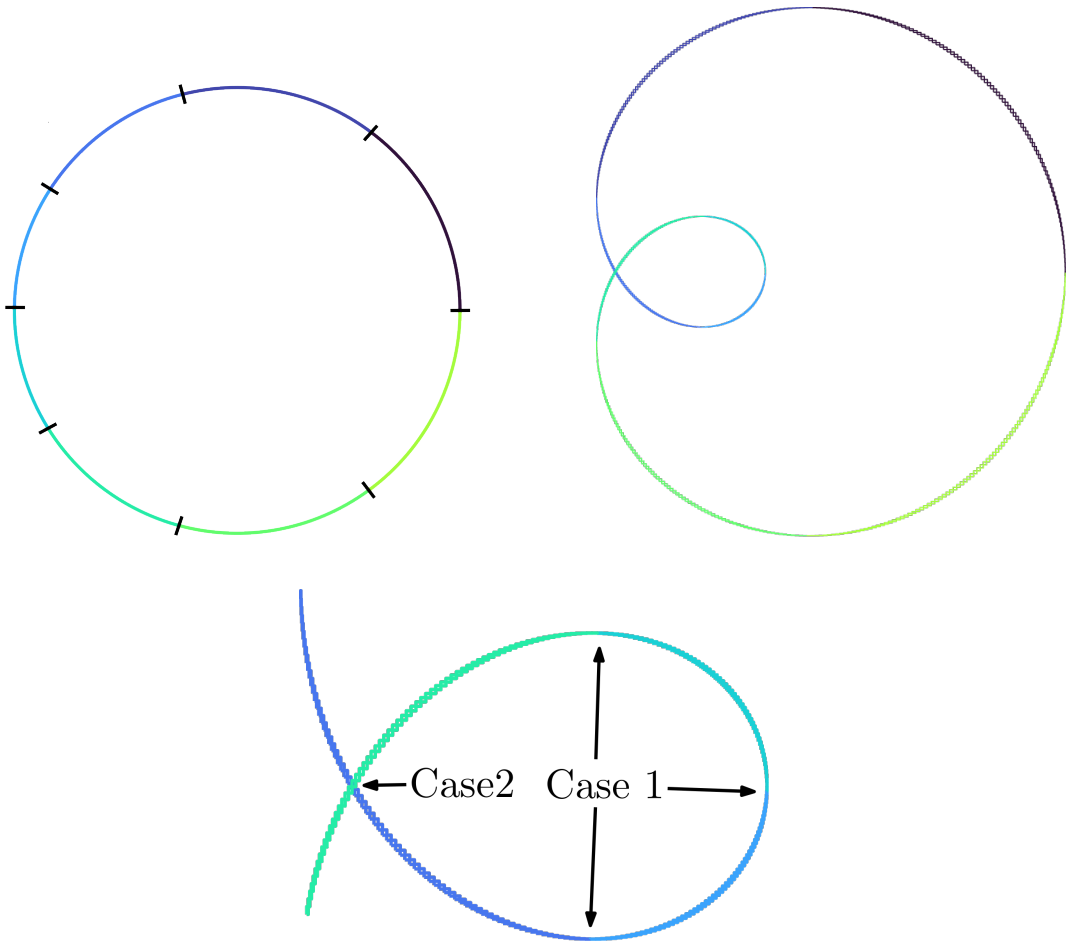
t_4 interesects t_3 ?



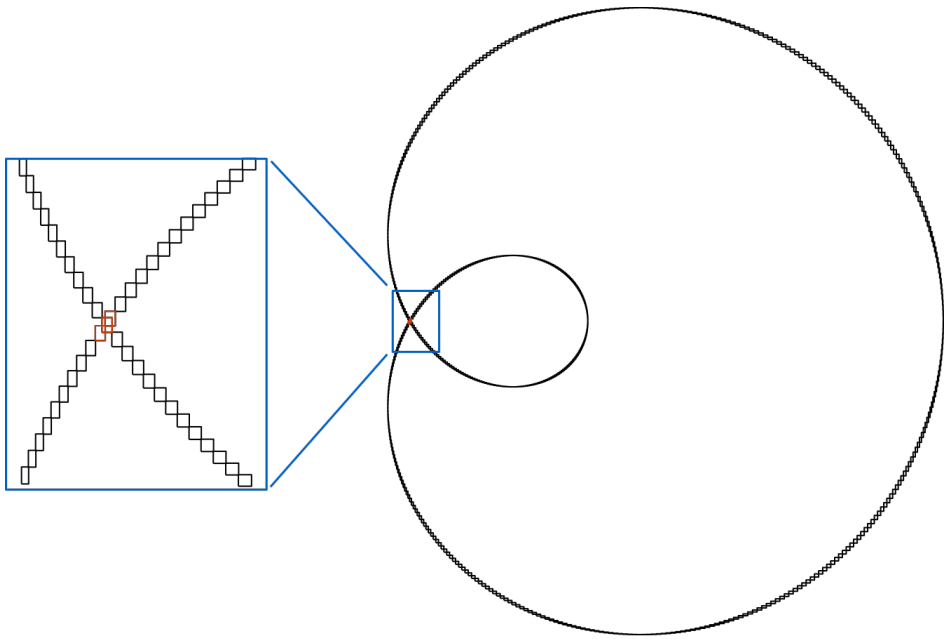

0
2π
t_1
t_2
t_3
t_4
t_5
t_6


BC_1
t_1
t_2
BC_2
t_3
t_4
t_5
BC_3
t_6

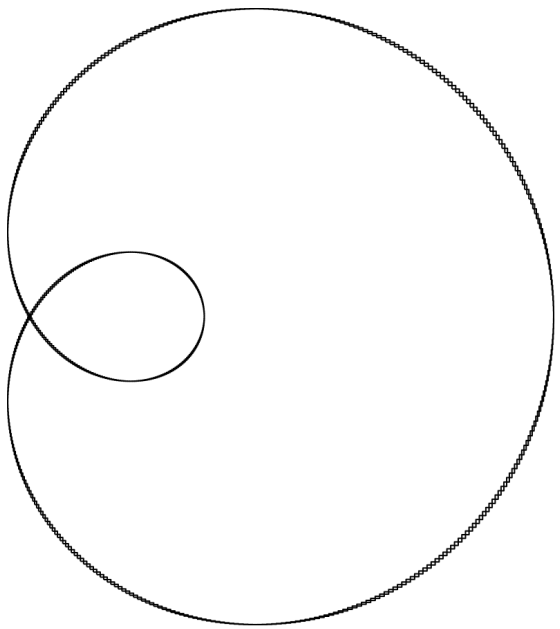
Box Chain decomposition result

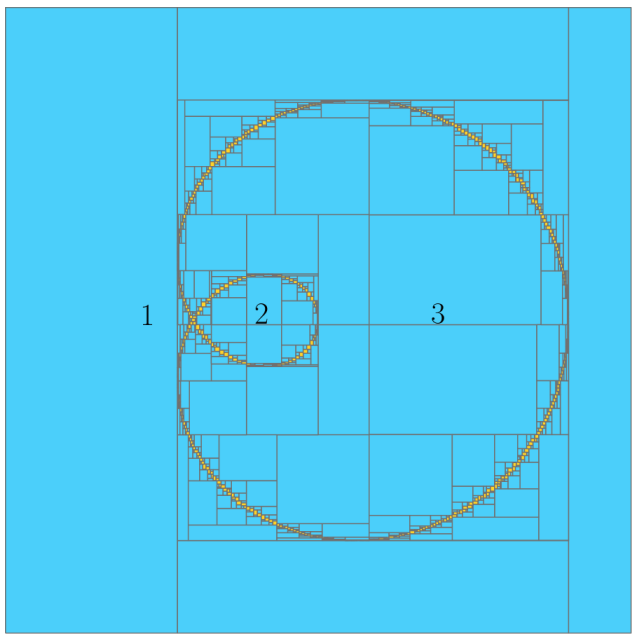


Real Intersections

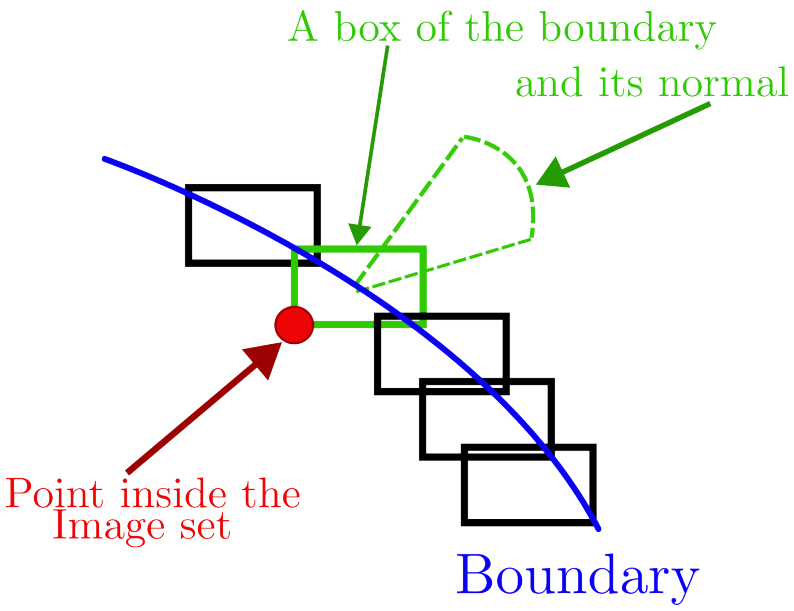


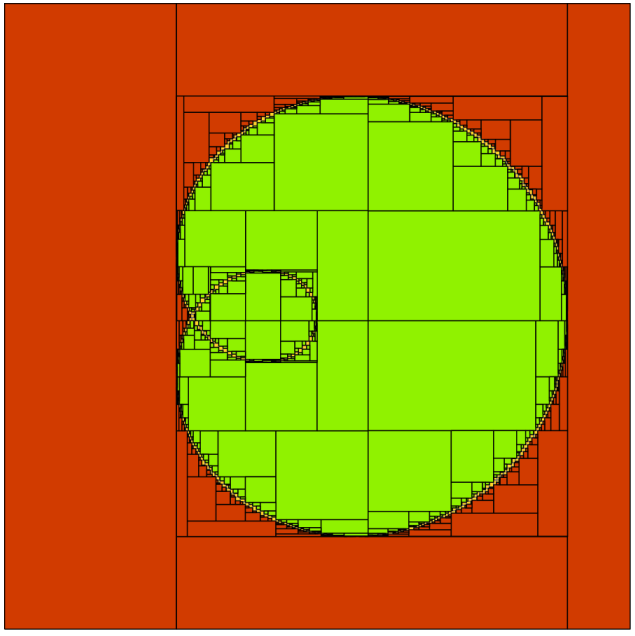
Initial paving



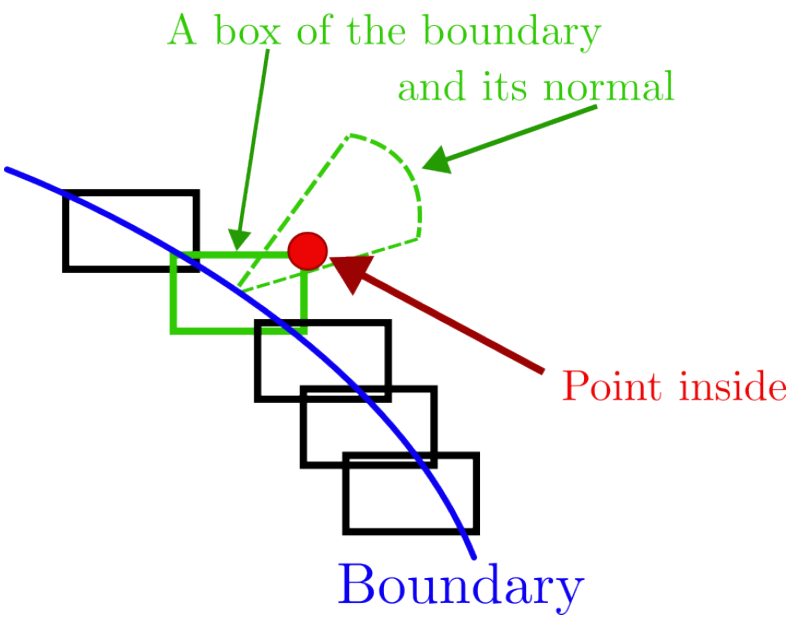


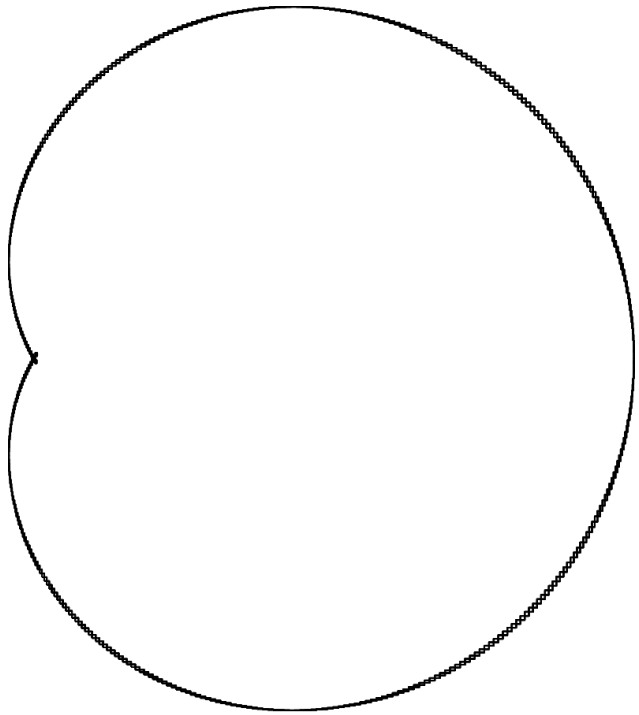
In areas


A box of the boundary
and its normal
Point inside the
Image set
Boundary

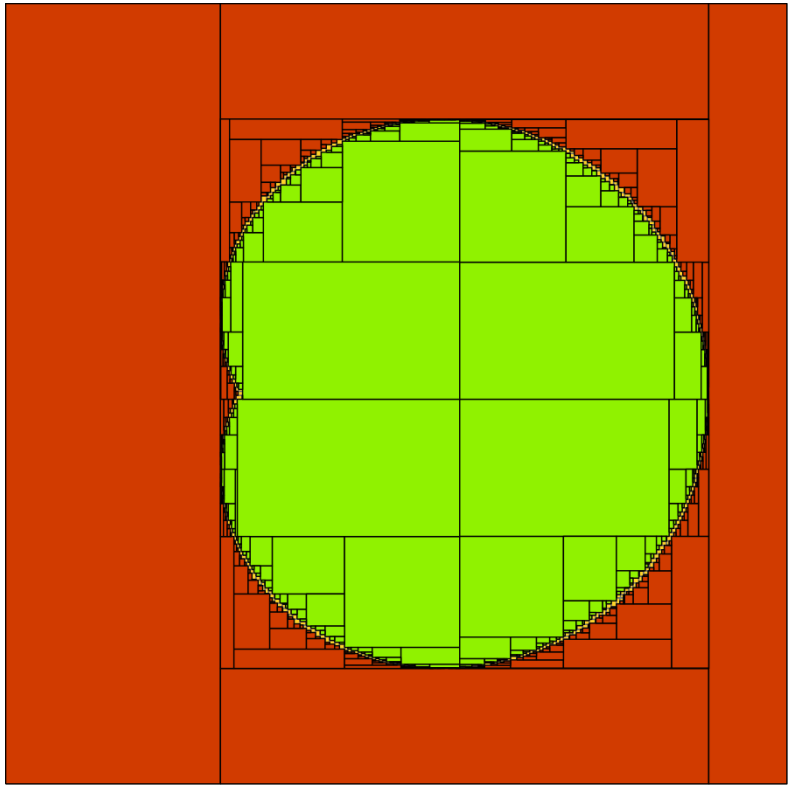


Removing fake boundaries

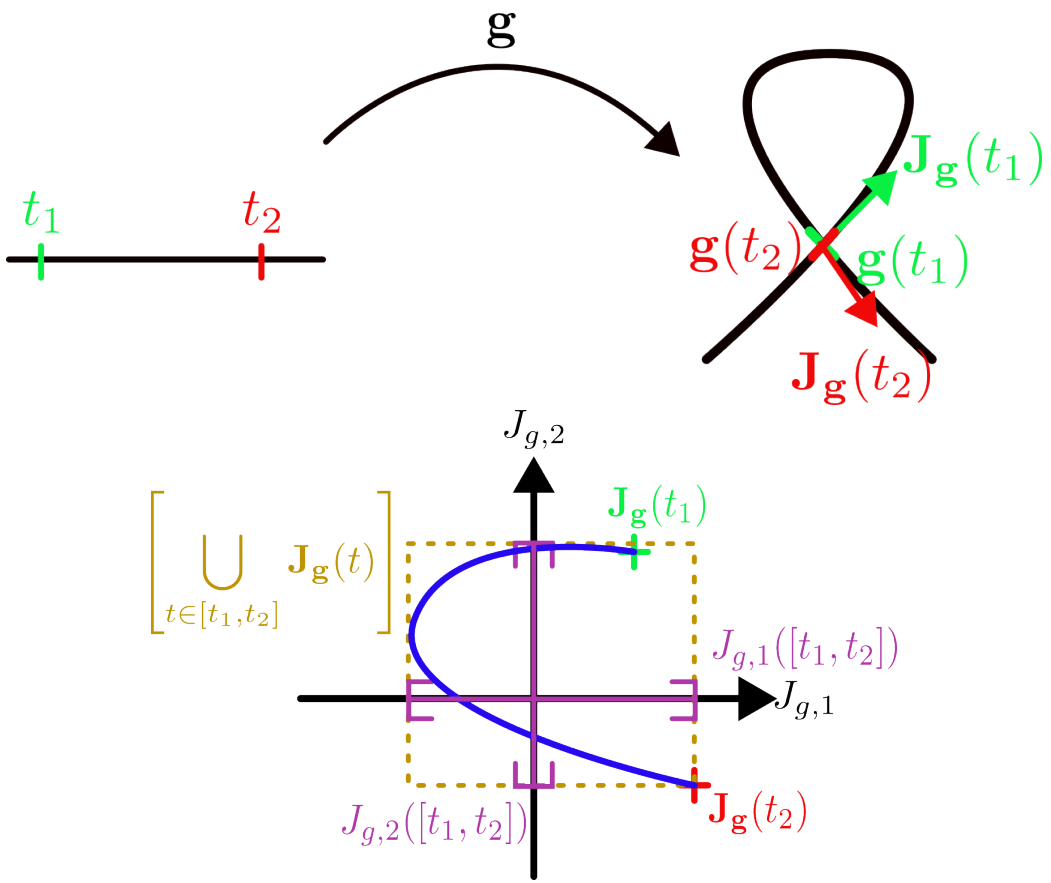




Better inner approximation



Injectivity criteria



2nd example