Interval Robotics
Chapter 2: Subpavings

L. Jaulin

Set inversion

Let $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbb{Y}$ be a subset of $\mathbb{R}^m$. Set inversion is the characterization of

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

We shall use the following tests;

$$\begin{array}{lllll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

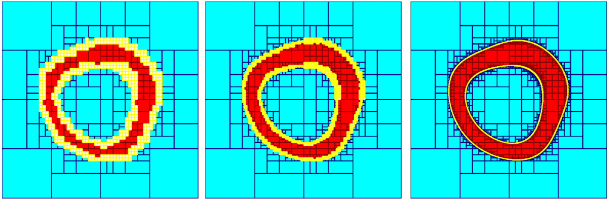
A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$. Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Exercise. The set

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}$$

are approximated by $\mathbb{X}^-$ and $\mathbb{X}^+$ for different accuracies. Denote by $\mathbb{R}, \mathbb{Y}, \mathbb{B}$ the union of red, yellow, blue boxes. Denote by $\partial\mathbb{X}$ the boundary of $\mathbb{X}$.



$\mathbb{X}^- \cap \mathbb{B} = \emptyset$	yes or no
$\mathbb{X} \cap \mathbb{B} \neq \emptyset$	yes or no
$\mathbb{X}^+ = \mathbb{R} \cup \mathbb{Y}$	yes or no
$\partial \mathbb{X} \supset \mathbb{Y}$	yes or no
$\mathbb{X} \backslash (\mathbb{R} \cup \mathbb{B}) = \mathbb{Y} \cap \mathbb{X}$	yes or no

Solution. We have

$$
\begin{array}{lll}
\mathbb{X}^- \cap \mathbb{B} = \emptyset & \rightarrow & \text{Yes} \\
\mathbb{X} \cap \mathbb{B} = \emptyset & \rightarrow & \text{No} \\
\mathbb{X}^+ = \mathbb{R} \cup \mathbb{Y} & \rightarrow & \text{Yes} \\
\partial \mathbb{X} \supset \mathbb{Y} & \rightarrow & \text{No. Instead, we have } \partial \mathbb{X} \subset \mathbb{Y} \\
\mathbb{X} \backslash (\mathbb{R} \cup \mathbb{B}) = \mathbb{Y} \cap \mathbb{X} & \rightarrow & \text{Yes}
\end{array}
$$

Set operations such as $\mathbb{Z} := \mathbb{X} + \mathbb{Y}$, $\mathbb{X} := \mathbf{f}^{-1}(\mathbb{Y}), \mathbb{Z} := \mathbb{X} \cap \mathbb{Y} \dots$ can be approximated by subpaving operations.

Exercise. Consider the set

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4\}.$$

Find two subpavings $\mathbb{X}^-$ and $\mathbb{X}^+$, both made with a single box, such that

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Solution. We can take

$$\mathbb{X}^{-} = \left\{ \left[-\sqrt{2}, \sqrt{2} \right]^{\times 2} \right\} \text{ and } \mathbb{X}^{+} = \left\{ [-2, 2]^{\times 2} \right\}.$$

Here, subpavings are singletons, but it is not the case usually.

If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{Y} \subset \mathbb{R}^m$.

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

Exercise. Define the set

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 x_2 + \sin x_2 \leq 0 \text{ and } x_1 - x_2 = 1\}.$$

Show that it is a set inversion problem.

Solution. We have

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

with

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 x_2 + \sin x_2 \\ x_1 - x_2 \end{pmatrix} \text{ and } \mathbb{Y} = [-\infty, 0] \times \{1\}.$$

$$\begin{array}{lllll} \text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

SIVIA

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Algorithm Sivia(in: $[\mathbf{x}](0), \mathbf{f}, \mathbb{Y}$)

1 $\mathscr{L} := \{[\mathbf{x}](0)\}$;
2 pull $[\mathbf{x}]$ from $\mathscr{L}$;
3 if $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$, draw($[\mathbf{x}]$,'red');
4 elseif $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$, draw($[\mathbf{x}]$,'blue');
5 elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$,'yellow')};
6 else bisect $[\mathbf{x}]$ and push into $\mathscr{L}$;
7 if $\mathscr{L} \neq \emptyset$, go to 2

If $\Delta\mathbb{X}$ denotes the union of yellow boxes and if $\mathbb{X}^-$ is the union of red boxes then :

$$\mathbb{X}^- \subset \mathbb{X} \subset \underbrace{\mathbb{X}^- \cup \Delta\mathbb{X}}_{\mathbb{X}^+}.$$

Stack-queue

A *queue* is a list on which two operations are allowed:

- add an element at the end (*push*)
- remove the first element (*pull*).

A *stack* is a list on which two operations are allowed:

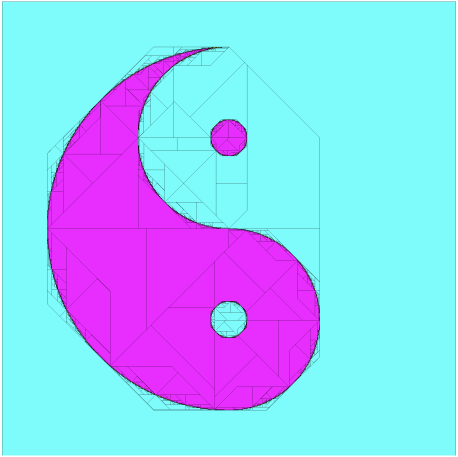
- add an element at the beginning of the list (*stack*)
- remove the first element (*pop*).

Example: Let $\mathscr{L}$ be an empty queue.

k	operation	result
0		$\mathscr{L} = \emptyset$
1	push$(\mathscr{L}, a)$	$\mathscr{L} = \{a\}$
2	push$(\mathscr{L}, b)$	$\mathscr{L} = \{a, b\}$
3	$x :=$ pull$(\mathscr{L})$	$x = a, \mathscr{L} = \{b\}$
4	$x :=$ pull$(\mathscr{L})$	$x = b, \mathscr{L} = \emptyset.$

If $\mathscr{L}$ is a stack, the table becomes

k	operation	result
0		$\mathscr{L} = \emptyset$
1	stack$(\mathscr{L}, a)$	$\mathscr{L} = \{a\}$
2	stack$(\mathscr{L}, b)$	$\mathscr{L} = \{a, b\}$
3	$x :=$ pop$(\mathscr{L})$	$x = b, \mathscr{L} = \{a\}$
4	$x :=$ pop$(\mathscr{L})$	$x = a, \mathscr{L} = \emptyset$.



Sivia with octogones (made by D. Massé)

Paver

A paver is an algorithm which generates boxes by bisections and classifies them.

Take

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid t(\mathbf{x}) = 1\} = t^{-1}(1)$$

We want an enclosure of the form

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+$$

Algorithm Sivia(in: $[\mathbf{x}](0), [t]$)
1 $\mathscr{L} := \{[\mathbf{x}](0)\}$;
2 pull $([\mathbf{x}], \mathscr{L})$;
3 if $[t]([\mathbf{x}]) = 1$, draw($[\mathbf{x}]$,'red');
4 elseif $[t]([\mathbf{x}]) = 0$, draw($[\mathbf{x}]$,'blue');
5 elseif $w([\mathbf{x}]) < \varepsilon$, {draw ($[\mathbf{x}]$,'yellow')};
6 else bisect $[\mathbf{x}]$ into $[\mathbf{x}](1)$ and $[\mathbf{x}](2)$; push $([\mathbf{x}](1), [\mathbf{x}](2), \mathscr{L})$;
7 if $\mathscr{L} \neq \emptyset$, go to 2

Define

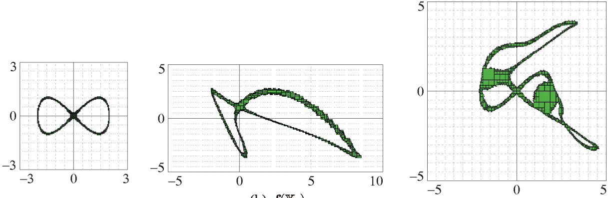
$$\mathbf{f}(x_1,x_2)=\begin{pmatrix} (x_1-1)^2-1+x_2 \\ -x_1^2+(x_2-1)^2 \end{pmatrix},$$

and

$$\mathbb{X}_1=\left\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^4-x_1^2+4x_2^2\in[-0.1,0.1]\right\}.$$

We shall compute $\mathbb{X}_1$, $\mathbf{f}(\mathbb{X}_1)$ and $\mathbf{f}^{-1}\circ\mathbf{f}(\mathbb{X}_1)$.

Image of a set



(a): $\mathbb{X}_1$ (b): $\mathbf{f}(\mathbb{X}_1)$ (c): $\mathbf{f}^{-1}(\mathbf{f}(\mathbb{X}_1))$

Sivia works with other abstract domains (or wrappers).

Application

Exercise. Consider a parabola of the form

$$\phi(\mathbf{p}, t) = p_1 t^2 + p_2 t + p_3.$$

where $\mathbf{p} = (p_1, p_2, p_3)^\mathsf{T}$ is an unknown parameter vector. Assume that

$$\phi(\mathbf{p}, 1) \in [2, 3], \ \phi(\mathbf{p}, 4) \in [5, 6], \ \phi(\mathbf{p}, 7) \in [8, 9].$$

Show that the set $\mathbb{P}$ of all feasible $\mathbf{p}$ can be defined as a set inversion problem.

Solution. We have

$$\mathbb{P} = \mathbf{f}^{-1}(\mathbb{Y}),$$

where

$$\mathbf{f}(\mathbf{p}) = \begin{pmatrix} \phi(\mathbf{p},1) \\ \phi(\mathbf{p},4) \\ \phi(\mathbf{p},7) \end{pmatrix} = \begin{pmatrix} p_1 + p_2 + p_3 \\ 16p_1 + 4p_2 t + p_3 \\ 49p_1 + 7p_2 + p_3 \end{pmatrix}$$

and

$$\mathbb{Y} = [2,3] \times [5,6] \times [8,9].$$

Model : $\phi(\mathbf{p},t) = p_1 e^{-p_2 t}$.
Prior feasible box for the parameters : $[\mathbf{p}] \subset \mathbb{R}^n$
Measurement times : $t_1, t_2, \ldots, t_m$
Data bars : $[y_1^-, y_1^+], [y_2^-, y_2^+], \ldots, [y_m^-, y_m^+]$

$$\mathbb{S} = \{\mathbf{p} \in [\mathbf{p}], \phi(\mathbf{p},t_1) \in [y_1^-, y_1^+], \ldots, \phi(\mathbf{p},t_m) \in [y_m^-, y_m^+]\}.$$

If

$$\phi(\mathbf{p}) = \begin{pmatrix} \phi(\mathbf{p}, t_1) \\ \\ \phi(\mathbf{p}, t_m) \end{pmatrix}$$

and

$$[\mathbf{y}] = [y_1^-, y_1^+] \times \cdots \times [y_m^-, y_m^+]$$

then

$$\mathbb{P} = [\mathbf{p}] \cap \phi^{-1}([\mathbf{y}]).$$

References

1. Interval analysis [3, 1, 2]
2. IAMOOC [2]

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