# Bisectable Abstract Domains

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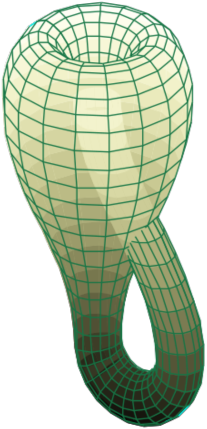
# Goal

Generalize interval algorithms with bisections.
Introduce *bisectable abstract domains* (or '*bad*' for short).
Introduce the *boxpies* as a specific *bad.*
Use boxpies to characterize the solution set of constraints involving complex numbers.

# What is a Bad ?

Consider a Riemannian manifold $\mathbb{M}$ such a $\mathbb{R}$, $\mathbb{R}^n$, a sphere, the Klein bottle, etc.



**Question** : Is such a paving always possible ? How to define the intersection, the union of the 'boxes'?

Denote by $d(a,b)$ the distance between $a$ and $b$.
We define the *diameter* $w(\mathbb{X})$, $\mathbb{X} \subset \mathbb{M}$.
A *bad* family $\mathbb{IM}$ is a family of subsets of $\mathbb{M}$ which satisfies some properties.

1) $\mathbb{IM}$ is a Moore family (containing $\mathbb{M}$), *i.e.,*

$$[a](1) \in \mathbb{IM}, [a](2) \in \mathbb{IM}, \ldots \Rightarrow \bigcap_i [a](i) \in \mathbb{IM}$$

Note that $(\mathbb{IM}, \subset)$ is a lattice but not a sublattice of $\mathscr{P}(\mathbb{M})$. Indeed:

$$\underbrace{[a] \cup [b]}_{\in \mathscr{P}(\mathbb{M})} \subset \underbrace{[a] \sqcup [b]}_{\in \mathbb{IM}}.$$

2) $\mathbb{IM}$ is equipped with a *bisector*, *i.e.*, a function $\beta : \mathbb{IM} \to \mathbb{IM} \times \mathbb{IM}$. If $\beta([x]) = \{[a], [b]\}$ :

*(i)* $[a]$ and $[b]$ do not overlap,

*(ii)* $[a]$ and $[b]$ cover $[x]$

(iii) $\beta$ minimizes $\max\{w([a]), w([b])\}$.

**Note**: For the implementation, the bisector is defined from a starting point: the *origin* (plane, tore, sphere).

**Question**: Is the set of boxes of $\mathbb{R}^n$ a *bad* ?

**Question**: Is any singleton of $\mathbb{M}$ a *bad* ?

**Question**: Can *bad* be defined when the Euler-Poincarré characteristic of $\mathbb{M}$ is non-zero?

**Answer**. Yes. Even if (once the bisector is defined) the poles yields implementation difficulties.

# Angles

Consider the equivalence relation on $\mathbb{R}$

$$\alpha \sim \beta \Leftrightarrow \frac{\beta - \alpha}{2\pi} \in \mathbb{Z}.$$

The set $\mathbb{A}$ of all angles is

$$\mathbb{A} = \frac{\mathbb{R}}{\sim} = \frac{\mathbb{R}}{2\pi\mathbb{N}}.$$

For simplicity, we will also write $\mathbb{A} = [-\pi, \pi]$.
Note that the set $\mathbb{A}$ is a Riemannian manifold.

If $\alpha$ and $\beta$ are angles and if $\rho \in \mathbb{R}$, we can define $\alpha + \beta$, $\alpha - \beta$ and $\rho \cdot \alpha$.

**Question**: Is $\mathbb{A}$ a vector space ?

**Answer**: No it is not. Indeed

$$\rho(\alpha+\beta) \neq \rho\alpha + \rho\beta.$$

Take for instance $\alpha = \beta = \pi$ and $\rho = \frac{1}{2}$.

**Question**: Is the set of angles $\mathbb{A}$ a lattice ?

**Answer**: No, due to its circular structure. It is thus not possible to define intervals of angles in order to apply interval techniques.

# Arcs

An *arc* $\langle\alpha\rangle$ is a connected subset of $\mathbb{A}$ . We have $\langle\alpha\rangle = \langle\overline{\alpha}, \widetilde{\alpha}\rangle$ with $\overline{\alpha} \in \mathbb{A}$ and $\widetilde{\alpha} \in [0, \pi]$.

The set of all arcs is denoted by $\mathbb{IA}$.

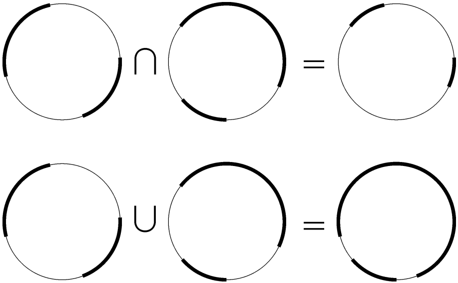
**Question**: Is $\mathbb{IA}$ is a Moore family ?

**Answer**: No. The intersection in $\mathbb{IA}$ is not closed.

**Question**: What is the smallest Moore family which contains $\mathbb{IA}$ ?

**Answer**: Unions of arcs.

A union of non overlapping arcs is called a *circular paving*.
The set of circular pavings is denoted by $\mathbb{UA}$ and $(\mathbb{UA}, \subset)$.



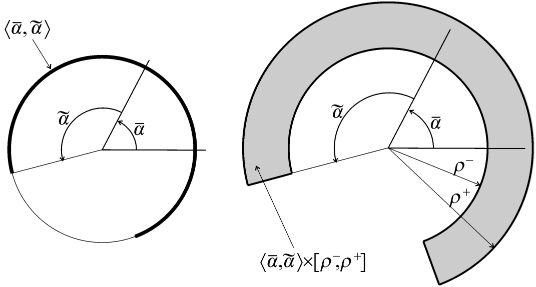
**Note**. It may be dangerous to deal with union of arcs.
**Example of Chabert**. With initial domains $[x] = [y] = [1, 9]$,

$$\begin{cases} y = x \\ 9\,(x-5)^2 = 16y \end{cases}$$

an explosion of the interval propagation occurs.

# Pies

The Cartesian product of *bads* is a *bad*.
A *pie* is an element of $\mathbb{UA} \times \mathbb{IR}$, i.e:
If $\alpha \in <\alpha>$ and $\rho \in [\rho]$ then the pair $(\alpha, \rho) \in <\alpha> \times [\rho]$ which is *pie*.




Left: an arc; Right: a pie with a single connected component

A pie can be denoted with a polar form: $[\rho]\, e^{i<\alpha>}$.
The intersection is closed:

$$[\rho_1]\, e^{i<\theta_1>} \cap [\rho_2]\, e^{i<\theta_2>} \;=\; ([\rho_1] \cap [\rho_2])\, e^{i(<\theta_1> \cap <\theta_2>)}.$$

# Boxpies

Both $\mathbb{IC}$ (the boxes of $\mathbb{C}$) and $\mathbb{UA} \times \mathbb{IR}$ (the pies) are Moore families in $\mathscr{P}(\mathbb{C})$.

Reduced product $\otimes$: $\mathbb{BP} = \mathbb{IC} \otimes \mathbb{UA} \times \mathbb{IR}$.

The family $\mathbb{BP}$ contains boxes and pies and all intersections between one box and one pie.

An element of $\mathbb{BP}$ is called a *boxpie*.

A boxpie can thus be written as

$$[x] + i\,[y] \;\cap\; [\rho]\,e^{i<\theta>}.$$

Note that the intersection in $\mathbb{BP}$ is closed:

$$\begin{aligned} & [x_1]+i\,[y_1] \;\cap\; [\rho_1]\,e^{i<\theta_1>} \;\cap\; [x_2]+i\,[y_2]\cap[\rho_2]\,e^{i<\theta_2>} \\ = \;& [x_1]\cap[x_2]+i\,([y_1]\cap[y_2])\cap([\rho_1]\cap[\rho_2])\,e^{i(<\theta_1>\cap<\theta_2>)}. \end{aligned}$$

**Why boxpies**? An arithmetic on boxpies inherits the good properties of $\mathbb{IC}$ for the addition, but also of good properties of $\mathbb{UA} \times \mathbb{IR}$ for the multiplication.

**Selfconsistency**. The expression for a boxpie may not be unique, e.g., the boxpie

$$[0,1]+i\,[1,2]\cap[1,2]\cdot e^{i[0,\frac{\pi}{4}]}=[1,1]+i\,[1,1]\cap[\sqrt{2},\sqrt{2}]e^{i\left[\frac{\pi}{4},\frac{\pi}{4}\right]}$$

is the singleton $1+i=\sqrt{2}e^{i\frac{\pi}{4}}$.

# Contractors

Denote by $\mathscr{L}$ *a* set of *bad*. A contractor is an operator

$$\mathscr{C} : \begin{array}{ccc} \mathscr{L} & \rightarrow & \mathscr{L} \\ \mathbb{X} & \mapsto & \mathscr{C}(\mathbb{X}) \end{array}$$

which satisfies

$$\begin{array}{ll} \mathbb{X} \subset \mathbb{Y} \Rightarrow \mathscr{C}(\mathbb{X}) \subset \mathscr{C}(\mathbb{Y}) & \text{(monotonicity)} \\ \mathscr{C}(\mathbb{X}) \subset \mathbb{X} & \text{(contractance)} \end{array}$$

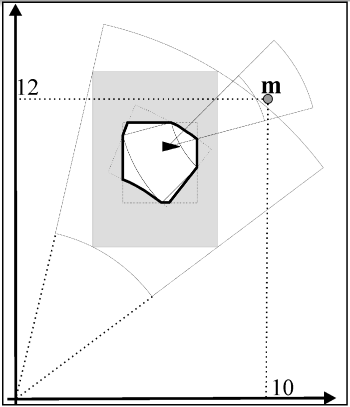
**Constraint propagation**. To each constraint $c_j \in \{c_1, \ldots, c_m\}$ of a constraint network, a contractor $\mathscr{C}_j(\mathbb{X})$ is built. We apply $\mathscr{C} = \mathscr{C}_1 \circ \cdots \circ \mathscr{C}_m$ until no more contraction can be observed.
**Separators**. A separator is a pair of two complementary contractors. Combined with a paver, separators makes it possible to compute an inner and an outer characterization of the solution set.

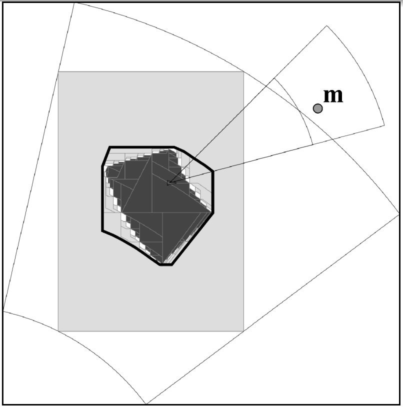
# Application

A robot, moving in a plane, is able to see a landmark $\mathbf{m}$ with coordinates $(10, 12)$.
More precisely, a sensor in the robot is able to measure the distance $d \in [4, 6]$ and the azimuth $\alpha \in [\frac{\pi}{12}, \frac{\pi}{6}]$ of $\mathbf{m}$.
We know that $\mathbf{m} \in [3, 8] \times [6, 13]$.

Let us represent the position of the robot by a complex number $p \in \mathbb{C}$. We have:

$$10 + 12i - p = de^{i\alpha}, p \in [3,8] \times [6,13],\ \alpha \in [\frac{\pi}{12}, \frac{\pi}{6}],\ d \in [4,6].$$



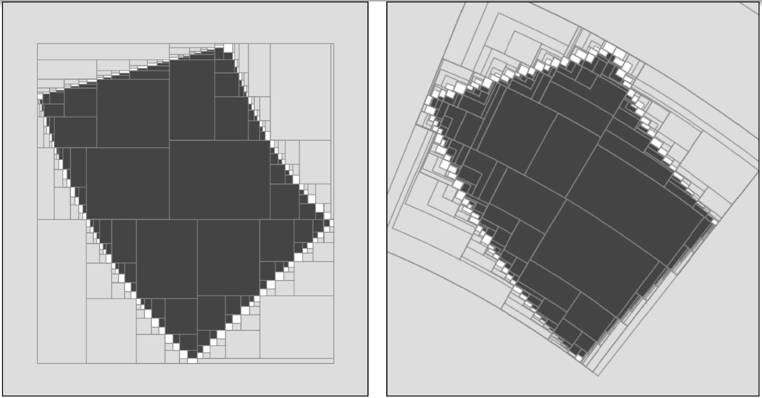




Left: first contraction; Right: Inner and outer approximation

To compare, we now consider boxes and pies as domains, but in a separate way.
We use some specific minimal separators for the projection of the set

$$\{(x, y, \rho, \theta) \mid x = \rho\cos\theta \text{ and } y = \rho\sin\theta\}$$

with respect to the $(x, y)$ and $(\rho, \theta)$ space.



Left: with boxes only; Right: with pies only