Luenberger contractor

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October 30 2024, Brest

1. Linear monotonic systems

Consider the system

$$\left\{ \begin{array}{rcl} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x} \end{array} \right.$$

The system is monotonic if the flow $\phi(t, \mathbf{x}_0)$ is monotonic in $\mathbf{x}_0$.

Proposition. If $\mathbf{A}$ is a Metzler matrix (*i.e.*, positive for all non diagonal elements), then the system is monotonic.
Proof. We have

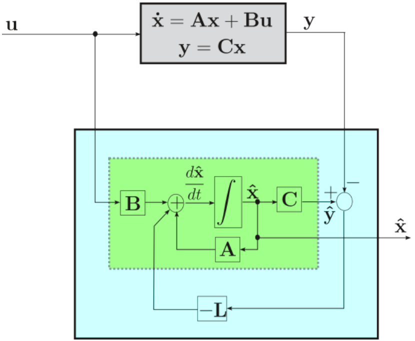
$$\begin{array}{rcl} \mathbf{x}(t+dt) & = & (\mathbf{I}+dt\cdot\mathbf{A})\cdot\mathbf{x}(t) \\ & = & \left(\begin{array}{cccc} 1+dt\cdot a_{11} & dt\cdot a_{12} & dt\cdot a_{13} & \cdots \\ dt\cdot a_{21} & 1+dt\cdot a_{22} & dt\cdot a_{23} & \cdots \\ dt\cdot a_{31} & dt\cdot a_{32} & 1+dt\cdot a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right)\cdot\mathbf{x}(t) \end{array}$$

which corresponds to a monotonic evolution as soon as $a_{ij} > 0$, for $i \neq j$.

2. Luenberger observer

Consider the system

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u} \\ \mathbf{y} & = & \mathbf{C}\mathbf{x} \end{cases}$$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We have:

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u} \\ \frac{d}{dt}\hat{\mathbf{x}} & = & \mathbf{A}\hat{\mathbf{x}}+\mathbf{B}\mathbf{u}-\mathbf{L}\left(\mathbf{C}\hat{\mathbf{x}}-\mathbf{y}\right) \end{cases}$$

Set $\varepsilon = \hat{\mathbf{x}} - \mathbf{x}$. We get:

$$\begin{array}{rcl} \frac{d}{dt}(\hat{\mathbf{x}} - \mathbf{x}) & = & \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} - \mathbf{L}(\mathbf{C}\hat{\mathbf{x}} - \mathbf{C}\mathbf{x}) - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{u} \\ & = & \mathbf{A}(\hat{\mathbf{x}} - \mathbf{x}) - \mathbf{L}\mathbf{C}(\hat{\mathbf{x}} - \mathbf{x}) \end{array}$$

i.e.,

$$\dot{\varepsilon} = (\mathbf{A} - \mathbf{L}\mathbf{C})\,\varepsilon$$

in which the control $\mathbf{u}$ is not involved.
The estimation error ε tends towards zero if all the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ have negative real parts.

3. Interval Luenberger contractor

Consider the system

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u} \\ \mathbf{y} & = & \mathbf{C}\mathbf{x} \end{cases}$$

We define the Luenberger observer

$$\frac{d}{dt}\hat{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} - \mathbf{L}\left(\mathbf{C}\hat{\mathbf{x}} - \mathbf{y}\right)$$

or equivalently

$$\hat{\mathbf{x}}(t) = \mathscr{L}\left(\hat{\mathbf{x}}_0, \mathbf{y}(t)\right)$$

Proposition. If the matrix $\mathbf{L}$ is chosen such that $\mathbf{A}-\mathbf{LC}$ is Metzler, we have

$$\left.\begin{array}{l}\mathbf{x}(0)\in[\hat{\mathbf{x}}^-(0),\hat{\mathbf{x}}^+(0)]\\ \hat{\mathbf{x}}^-(t)=\mathscr{L}(\hat{\mathbf{x}}^-(0),\mathbf{y})\\ \hat{\mathbf{x}}^+(t)=\mathscr{L}(\hat{\mathbf{x}}^+(0),\mathbf{y})\end{array}\right\}\Rightarrow\forall t,\mathbf{x}(t)\in[\hat{\mathbf{x}}^-(t),\hat{\mathbf{x}}^+(t)]$$

Moreover, if $\mathbf{A}-\mathbf{LC}$ is Hurwitz, $\|\hat{\mathbf{x}}^+(t)-\hat{\mathbf{x}}^-(t)\|\to 0$.

Consequence: The application

$$[\mathbf{x}^-(t),\mathbf{x}^+(t)] \mapsto [\mathbf{x}^-(t),\mathbf{x}^+(t)] \cap [\mathscr{L}(\mathbf{x}^-(0),\mathbf{y}),\mathscr{L}(\mathbf{x}^+(0),\mathbf{y})]$$

is a tube contrator for the constraint

$$\left\{ \begin{array}{ccc} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u} \\ \mathbf{y} & = & \mathbf{C}\mathbf{x} \end{array} \right.$$

We thus get the forward tube contractor.

We define the backward system

$$\begin{cases} \dot{\mathbf{x}} & = & -\mathbf{A}\mathbf{x}-\mathbf{B}\mathbf{u} \\ \mathbf{y} & = & \mathbf{C}\mathbf{x} \end{cases}$$

From this backward system, we get the backward tube contractor.

We alternate the forward tube contractor and the backward tube contractor until the fixed point is reached.
It corresponds to the interval Luenberger contractor.

Example

Consider the system

$$\left\{ \begin{array}{lcl} \dot{\mathbf{x}} & = & \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{u} \\ \mathbf{y} & = & \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \end{array} \right.$$

We have

$$\mathbf{A} - \mathbf{LC} = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -\ell_1 \\ 1 & -\ell_2 \end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned} P(s) &= \det\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -2 & -\ell_1 \\ 1 & -\ell_2 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} s+2 & \ell_1 \\ -1 & s+\ell_2 \end{pmatrix} \\ &= (s+2)\cdot(s+\ell_2)+\ell_1 \\ &= s^2+(\ell_2+2)\,s+2\ell_2+\ell_1 \end{aligned}$$

The Routh table is

$$\begin{array}{cc} 1 & 2\ell_2+\ell_1 \\ \ell_2+2 & 0 \\ 2\ell_2+\ell_1 & \end{array}$$

The error ε converges to 0 and $\mathbf{A}-\mathbf{LC}$ is Metzler if

$$\begin{array}{ccc} \ell_2+2 & \geq & 0 \\ 2\ell_2+\ell_1 & \geq & 0 \\ -\ell_1 & \geq & 0 \end{array}$$

We can take

$$\ell_1 = -1, \ell_2 = 1$$

4. Ellipsoidal Luenberger contractor

Consider the system

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} & = & \mathbf{C}\mathbf{x} \end{cases}$$

with the Luenberger observer

$$\hat{\mathbf{x}}(t) = \mathscr{L}\left(\hat{\mathbf{x}}_0, \mathbf{y}(t)\right)$$

Proposition. A matrix $\mathbf{L}$ is chosen such that $\mathbf{A}-\mathbf{LC}$ is Hurwitz. Take a Lyapunov function $V(\varepsilon)=\varepsilon^T\mathbf{P}\varepsilon$ for the observation error ε such that

$$\dot{V}(\varepsilon)\leq -V(\varepsilon)\ .$$

We have

$$\forall t, V(\varepsilon(t))\leq e^{-t}V(\varepsilon(0))$$

and

$$\left.\begin{array}{c}\mathbf{x}(0)\in\mathscr{E}(\hat{\mathbf{x}}(0),\mathbf{P})\\ \hat{\mathbf{x}}(t)=\mathscr{L}(\hat{\mathbf{x}}(0),\mathbf{y})\end{array}\right\}\Rightarrow\forall t,\mathbf{x}(t)\in\mathscr{E}(\hat{\mathbf{x}}(t),e^{-t}\mathbf{P})$$

References

1. Luenberger observer [5] [3]
2. Lyapunov method [6]
3. Interval Luenberger [2][1][4]

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