Swim with brackets

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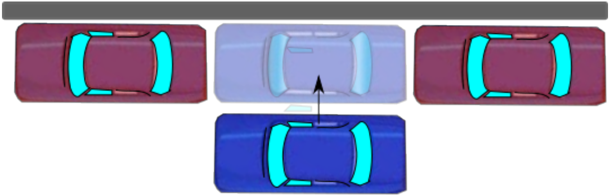
1. Control with Lie brackets



To park, the blue car needs to move sideway

$$\begin{cases} \dot{x}_1 &= u_1 \cos x_3 \\ \dot{x}_2 &= u_1 \sin x_3 \\ \dot{x}_3 &= u_2 \end{cases}$$

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot u_2$$

The Lie bracket between the two vector fields $\mathbf{f}$ and $\mathbf{g}$ is

$$[\mathbf{f},\mathbf{g}] = \frac{d\mathbf{g}}{d\mathbf{x}} \cdot \mathbf{f} - \frac{d\mathbf{f}}{d\mathbf{x}} \cdot \mathbf{g}.$$

Consider the system

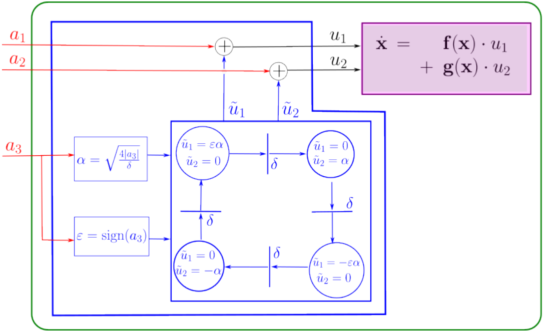
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \cdot u_1 + \mathbf{g}(\mathbf{x}) \cdot u_2.$$

Apply the following cyclic sequence:

$$\begin{array}{cccccc} t \in [0,\delta] & t \in [\delta, 2\delta] & t \in [2\delta, 3\delta] & t \in [3\delta, 4\delta] & t \in [4\delta, 5\delta] & \ldots \\ \mathbf{u} = (1,0) & \mathbf{u} = (0,1) & \mathbf{u} = (-1,0) & \mathbf{u} = (0,-1) & \mathbf{u} = (1,0) & \ldots \end{array}$$

where $\delta = o(1)$. We have (See Lavalle, Section 15.4.2.3.)

$$\mathbf{x}(t+4\delta) = \mathbf{x}(t) + [\mathbf{f},\mathbf{g}](\mathbf{x}(t))\,\delta^2 + o(\delta^2).$$



$\Leftrightarrow$

$$\begin{aligned}\dot{\mathbf{x}} = \; & \mathbf{f}(\mathbf{x}) \cdot a_1 \\ + \; & \mathbf{g}(\mathbf{x}) \cdot a_2 \\ + \; & [\mathbf{f}, \mathbf{g}](\mathbf{x}) \cdot a_3\end{aligned}$$

Equivalently, we have the system

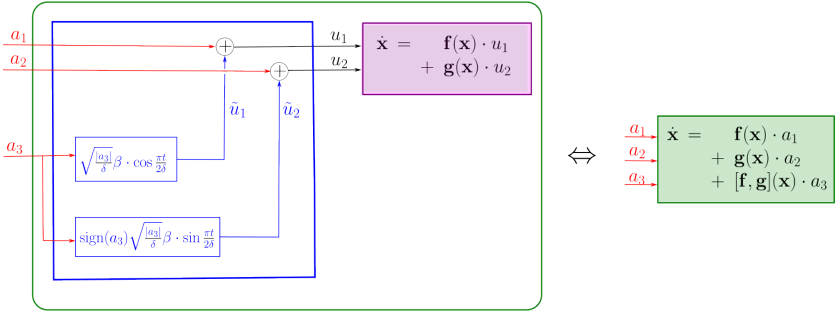
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \cdot u_1 + \mathbf{g}(\mathbf{x}) \cdot u_2.$$

We apply:

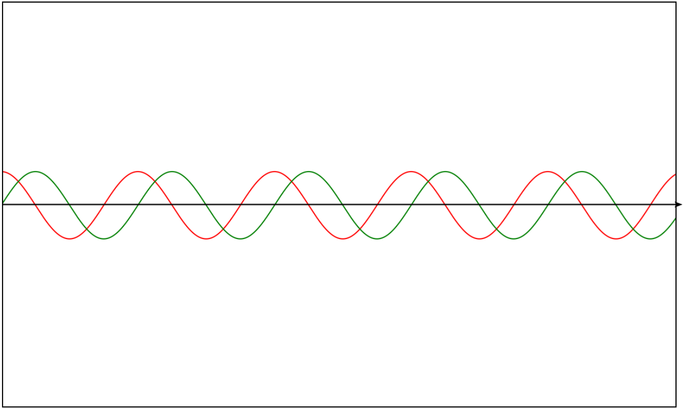
$$\mathbf{u} \propto \begin{pmatrix} \cos \frac{\pi t}{2\delta} \\ \sin \frac{\pi t}{2\delta} \end{pmatrix}$$

where $\delta = o(1)$. We have

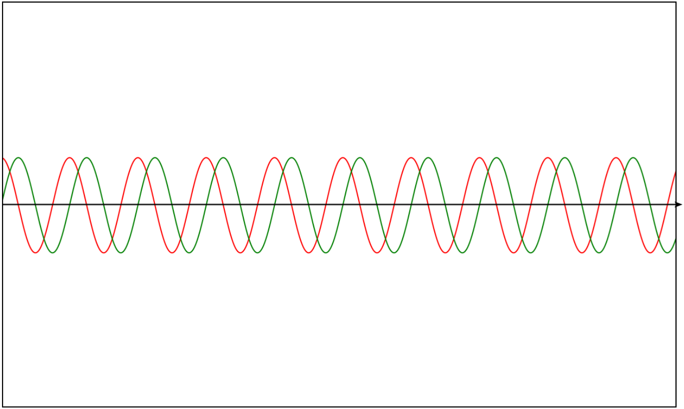
$$\mathbf{x}(t+4\delta) = \mathbf{x}(t) + [\mathbf{f}, \mathbf{g}](\mathbf{x}(t))\,\delta^2 + o(\delta^2).$$


a_1
a_2
a_3
u_1
u_2
$\tilde{u}_1$
$\tilde{u}_2$
$\sqrt{\frac{|a_3|}{\delta}}\beta \cdot \cos\frac{\pi t}{2\delta}$
$\mathrm{sign}(a_3)\sqrt{\frac{|a_3|}{\delta}}\beta \cdot \sin\frac{\pi t}{2\delta}$
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \cdot u_1 + \mathbf{g}(\mathbf{x}) \cdot u_2$
$\Leftrightarrow$
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \cdot a_1 + \mathbf{g}(\mathbf{x}) \cdot a_2 + [\mathbf{f}, \mathbf{g}](\mathbf{x}) \cdot a_3$

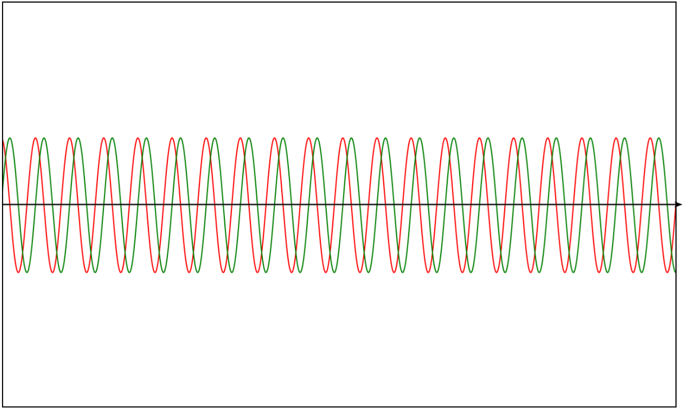
Joint distribution theory

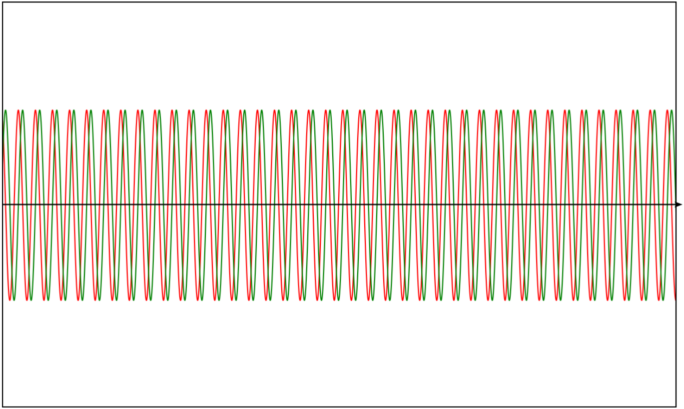


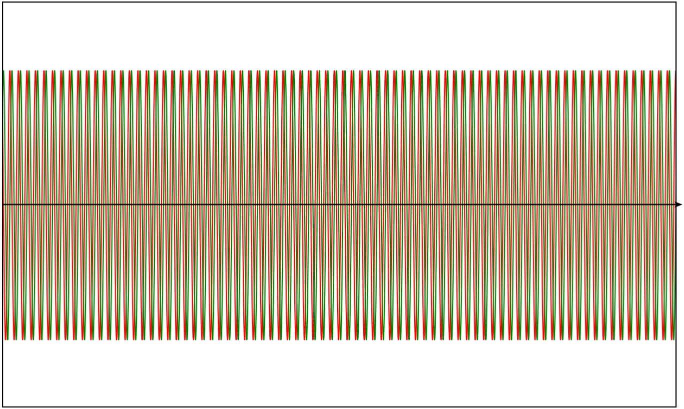
$u_1(t)$ and $u_2(t)$



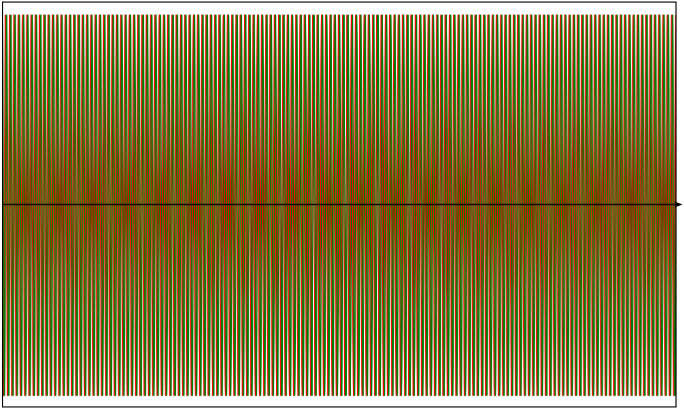
$$u_1 \cdot \int u_2 \geq 0$$







We converge to a strange distribution



What it this distribution?

Remark. Since $\delta = o(1)$, we cannot go fast along $[\mathbf{f}, \mathbf{g}]$.

For our Dubins car:

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot u_2$$

We have

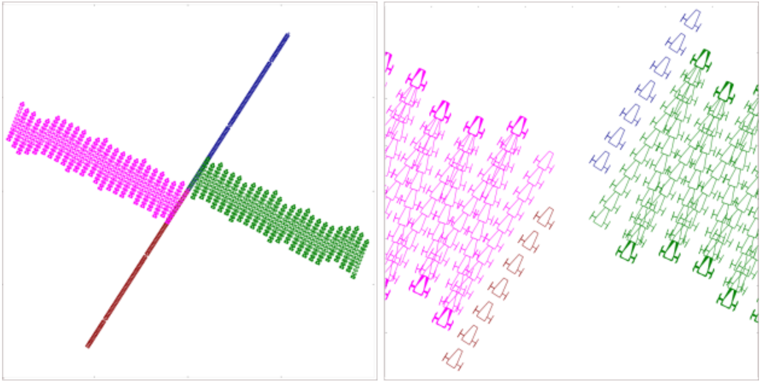
$$[\mathbf{f},\mathbf{g}](\mathbf{x}) = \underbrace{\frac{d\mathbf{g}}{d\mathbf{x}}(\mathbf{x})}_{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \cdot \underbrace{\mathbf{f}(\mathbf{x})}_{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}} - \underbrace{\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x})}_{\begin{pmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{pmatrix}} \cdot \underbrace{\mathbf{g}(\mathbf{x})}_{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{pmatrix}$$

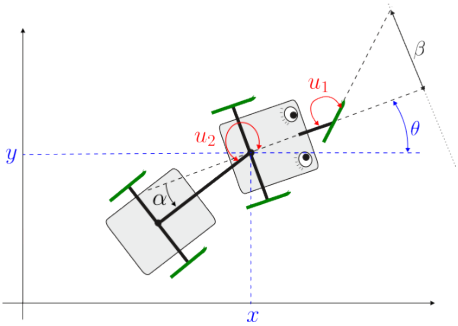
We can now move the car laterally.

If we apply the cyclic sequence, we get

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot a_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot a_2 + \underbrace{\begin{pmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{pmatrix}}_{[\mathbf{f},\mathbf{g}](\mathbf{x})} \cdot a_3$$



2. Skate car


β
u_1
u_2
θ
y
α
x

Model for the skate car

$$\begin{cases} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= v\beta \\ \dot{v} &= -(\beta+\sin\alpha)\,u_2 \\ \dot{\alpha} &= -v\,(\beta+\sin\alpha) \\ \dot{\beta} &= u_1 \end{cases}$$

Velocity model

$$\left\{\begin{array}{lcl}\dot{\theta} & = & v\beta \\ \dot{v} & = & -(\beta+\sin\alpha)\,u_2 \\ \dot{\alpha} & = & -v\,(\beta+\sin\alpha) \\ \dot{\beta} & = & u_1\end{array}\right.$$

$$\begin{pmatrix} \dot{\theta} \\ \dot{v} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} v\beta \\ 0 \\ -v(\beta + \sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ -\beta - \sin\alpha \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{g}_2} \cdot u_2$$

We want to control v to equal to $\bar{v}$. Since

$$\dot{v} = -(\beta + \sin\delta)\, u_2.$$

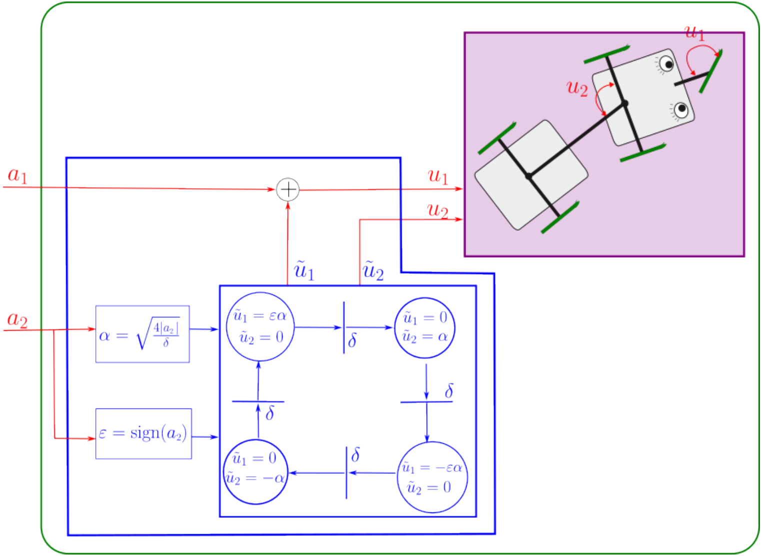
We could take

$$u_2 = -\frac{\bar{v} - v}{\beta + \sin\delta}$$

We get $\dot{v} = \bar{v} - v$. We quickly reach a singularity when $\beta + \sin\delta = 0$.

We have

$$
\begin{aligned}
[\mathbf{g}_1, \mathbf{g}_2] &= \frac{d\mathbf{g}_2}{d\mathbf{x}} \cdot \mathbf{g}_1 - \underbrace{\frac{d\mathbf{g}_1}{d\mathbf{x}}}_{=\mathbf{0}} \cdot \mathbf{g}_2 \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos\alpha & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$


a_1
a_2
u_1
u_2
$\tilde{u}_1$
$\tilde{u}_2$
$\alpha = \sqrt{\frac{4|a_2|}{\delta}}$
$\varepsilon = \text{sign}(a_2)$
$\tilde{u}_1 = \varepsilon\alpha$
$\tilde{u}_2 = 0$
$\tilde{u}_1 = 0$
$\tilde{u}_2 = \alpha$
$\tilde{u}_1 = -\varepsilon\alpha$
$\tilde{u}_2 = 0$
$\tilde{u}_1 = 0$
$\tilde{u}_2 = -\alpha$
δ

We now have the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\delta} \\ \dot{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} v\cos\theta \\ v\sin\theta \\ v\beta \\ 0 \\ -v(\beta+\sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1} \cdot a_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{[\mathbf{g}_1,\mathbf{g}_2]} \cdot a_2$$

Assume that we want to control v and β. i.e.

$$\begin{pmatrix} \dot{v} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{a}$$

We want

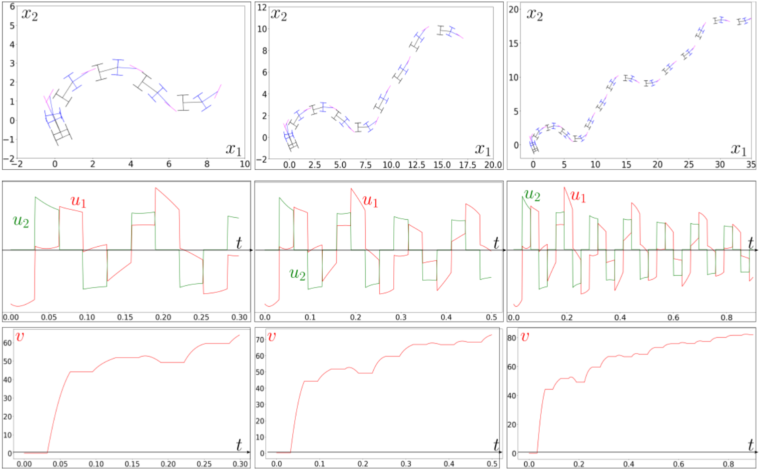
$$\begin{pmatrix} \dot{v} \\ \dot{\beta} \end{pmatrix} = K \begin{pmatrix} \bar{v} - v \\ \bar{\beta} - \beta \end{pmatrix}$$

Thus we take

$$\mathbf{a} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \left(K \begin{pmatrix} \bar{v} - v \\ \bar{\beta} - \beta \end{pmatrix} \right)$$

To have a heading control, we take

$$\dot{\bar{\beta}} = -0.01\bar{\beta} + 0.1(\bar{\theta} - \theta).$$



Desired heading $\bar{\theta} = \frac{\pi}{6}$ and a desired speed $\bar{v} = 100$.

3. Abstract swimmer

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}_1(\mathbf{x}) \cdot u_1 + \mathbf{g}_2(\mathbf{x}) \cdot u_2.$$

with

$$\mathbf{x} = (v, \cdots)$$

v driftless, i.e.,$f_1(\mathbf{x}) = 0$

$$[\mathbf{g}_1, \mathbf{g}_2] \parallel (1, 0, 0, \ldots)$$

The skate car:

$$\begin{pmatrix} \dot v \\ \dot x \\ \dot y \\ \dot\theta \\ \dot\alpha \\ \dot\beta \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ v\cos\theta \\ v\sin\theta \\ v\beta \\ -v(\beta+\sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1} \cdot u_1 + \underbrace{\begin{pmatrix} -(\beta+\sin\alpha) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{g}_2} \cdot u_2$$

with

$$[\mathbf{g}_1, \mathbf{g}_2] = (-1, 0, 0, \dots)$$

References

1. Lie bracket control [4] [3] [5]
2. Skate car [2]
3. Swimming robots [7][6][1]

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