Outer approximation of the occupancy set left by a mobile robot

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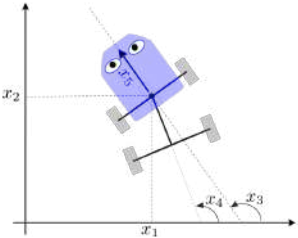
Occupancy set

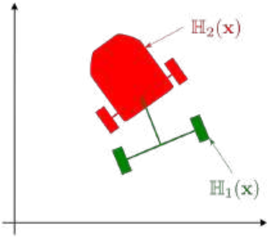
We have

- a multi-body mobile robot $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- an uncertain input $\mathbf{u}(t) \in [\mathbf{u}]$
- $\mathbb{H}_i(\mathbf{x})$ is the shape function of the ith body,

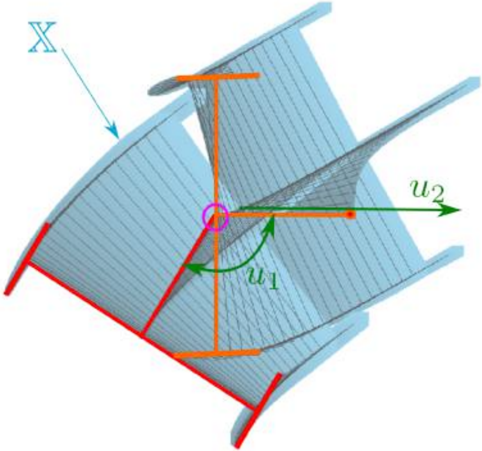
The *occupancy set* is

$$\mathbb{X} = \bigcup_{t \in [0, t_{\max}]} \bigcup_{i} \mathbb{H}_i(\mathbf{x}(t))$$





$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$


$\mathbb{X}$
u_2
u_1

Boatbot towing a magnetometer

1. Differential algebra

Complex numbers

Consider the set

$$\mathbb{R} < i > = \left\{ 1, 3, 3.1, \pi, \ldots, i, i^2, 1 + 2i + 5i^2, \frac{1}{1+i^5}, \ldots \right\}$$

Take the equation $i^2 + 1 = 0$. The quotient

$$\frac{\mathbb{R} < i >}{i^2 + 1} = \mathbb{C}$$

In $\mathbb{R} < i >$, i is algebraically independent.
In $\mathbb{C}$, i is algebraic (*e.g.*, $i^4 = 1$).
$\mathbb{C}/\mathbb{R}$ is a field extension.

Differential algebra

A differential ring is a ring $(\mathscr{R}, +, \cdot)$ equipped with the derivative $\frac{d}{dt}$

$$\begin{array}{ll} \text{(i)} & \mathbb{R} \subset \mathscr{R} \\ \text{(ii)} & \forall a \in \mathscr{R}, \frac{d}{dt}a \in \mathscr{R} \end{array}$$

where $\mathbb{R}$ is the set of real numbers.

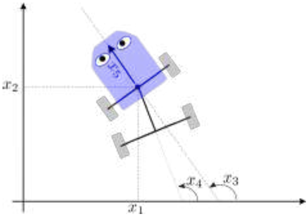
Moreover, $\frac{d}{dt}$ satisfies the classical rules. For instance

$$\begin{aligned}\frac{d}{dt}(a+b) &= \frac{d}{dt}a + \frac{d}{dt}b\\ \frac{d}{dt}(a\cdot b) &= \frac{d}{dt}a\cdot b + a\cdot\frac{d}{dt}b\end{aligned}$$

A system is finitely generated differential extension.
For instance, the system $\mathscr{S} : \dot{x} = x + u$ corresponds to the differential extension

$$\mathscr{S} : \frac{\mathbb{R} < u, x >}{\dot{x} - x - u}$$

Elimination methods (mainly differential Gröbner bases or resultant) are used in this context



$$\mathscr{S} : \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix} ; \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The variable x_i of the system is observable if $x_i \in \mathbb{R} < y_1, y_2 >$

We have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Thus $x_1 \in \mathbb{R} < y_1, y_2 >$ and $x_2 \in \mathbb{R} < y_1, y_2 >$.

We have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \end{pmatrix}$$

Thus $x_3 = \text{atan2}(\dot{x}_2, \dot{x}_1)$ and $x_5 = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$.
Thus $x_3 \in \mathbb{R} < y_1, y_2 >$ and $x_5 \in \mathbb{R} < y_1, y_2 >$.
We can show that $x_4 \notin \mathbb{R} < y_1, y_2 >$

2. Integral algebra

An *integral ring* is a a ring $(\mathscr{R},+,\cdot)$ equipped with the integration $\int$ such that

(i)	$\mathbb{R} \subset \mathscr{R}$
(ii)	$\forall a \in \mathscr{R}, \int a \in \mathscr{R}$

The meaning of $\int$ is the primitive which cancel for $t = 0$, i.e.,

$$\int a = \int_0^t a(\tau)d\tau.$$

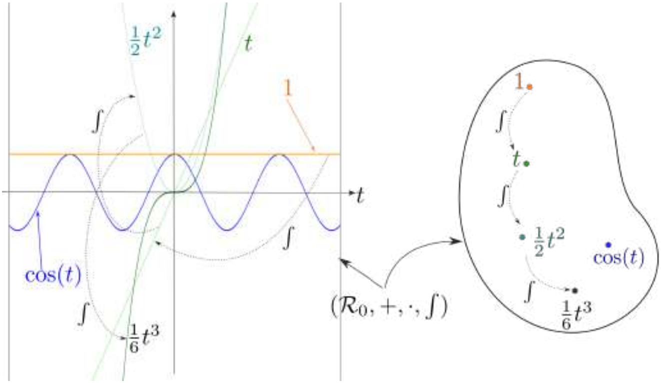
Moreover, $\int$ satisfies the classical integral rules. For instance

$$\begin{aligned} \int(a+b) &= \int a + \int b \\ \int a \cdot \int b &= \int\left(a \cdot \int b + \int a \cdot b\right) \end{aligned}$$

Consider $\mathscr{R}_0$ the smallest real integral ring.
We have

$$\begin{aligned} a = 2 \in \mathscr{R}_0 & \quad \text{it is a constant} \\ b = 2t \in \mathscr{R}_0 & \quad \text{since, } b = \int a \end{aligned}$$

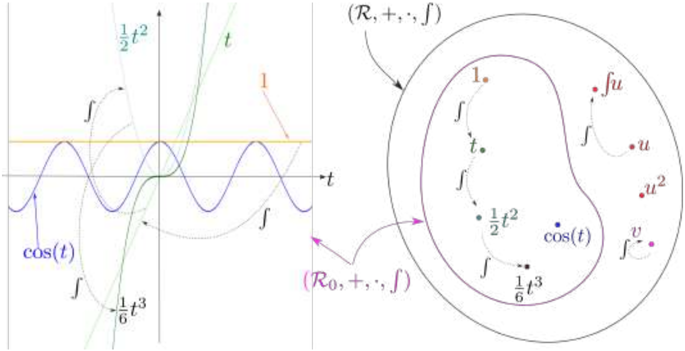
We assume that $(\mathscr{R}_0, +)$ is a topological group (*i.e.*, an infinite number of addition is allowed).


$\frac{1}{2}t^2$
t
1
$\int$
t
cos(t)
$\frac{1}{6}t^3$
$(\mathcal{R}_0, +, \cdot, \int)$
1
$\frac{1}{2}t^2$
cos(t)
$\frac{1}{6}t^3$

Consider an *integral ring extension* is $\mathscr{L}/\mathscr{R}$.
An element u of $\mathscr{L}$ is said to be integral $\mathscr{R}$-algebraic independent if

$$u, \int u, \int^2 u, \int^3 u, \ldots$$

are all independent.


$\frac{1}{2}t^2$
t
1
$\int$
t
$\cos(t)$
$\frac{1}{6}t^3$
$(\mathcal{R}, +, \cdot, \int)$
$(\mathcal{R}_0, +, \cdot, \int)$
1
t
$\frac{1}{2}t^2$
$\cos(t)$
$\frac{1}{6}t^3$
$\int u$
u
u^2
v

Integral dynamical system

Given an integral ring $\mathscr{R}$.
We denote by $\mathscr{R} < u_1, u_2, \cdots >$ the integral ring generated by $\mathscr{R}$ and by a finite set $\{u_1, u_2, \dots\}$ that are integral $\mathscr{R}$-algebraic independent.

Example. Consider the integral ring $\mathscr{L} = \mathscr{R}_0 < u >$. We have

$$\begin{array}{rcl} \cos t & \in & \mathscr{L} \\ u + \int \sin u + 3 & \in & \mathscr{L} \\ u + \int \left(\sin \int^3 u \right) + 3 & \in & \mathscr{L} \end{array}$$

Definition. An *integral dynamical system* is defined as a finite subset $\{x_1, \ldots, x_n\}$ of $\mathscr{R}_0 < u_1, \ldots, u_m >$.
$x_1, x_2, \ldots$ are called the state variables
$u_1, u_2, \ldots$ are called the inputs.

Consider a system of the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Equivalently,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{\tau=0}^{t} \mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau)) d\tau.$$

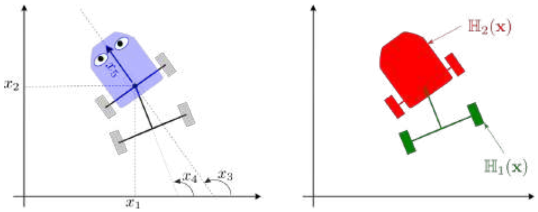
This system is an *integral dynamical system* if for all $i \in \{1, \ldots, n\}$, $x_i \in \mathscr{R}_0 < u_1, \ldots, u_m >$.

Interval extension of an integral dynamical system

Consider an integral dynamical system
$\{x_1 \ldots, x_n\} \in \mathscr{R}_0 < u_1, \ldots, u_m >$.

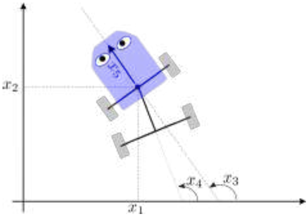
- For each x_i, we can build an expression which involves $x_1(0), \ldots, x_n(0)$, $u_1, \ldots, u_m$ as variables and $+, -, \cdot, /, \int$ as operators.
- An interval evaluation can be performed using the classical rules of interval arithmetic.
- We get an interval expression which provides a guaranteed enclosure for the trajectories $x_i(t)$.

Occupancy set

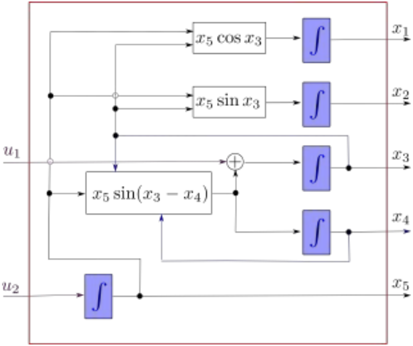


$$\begin{aligned} \mathbb{X} &= \bigcup_{t\in[0,t_{\max}]} \bigcup_i \mathbb{H}_i(\mathbf{x}(t)) \\ &\subset \bigcup_{[t_k]} \bigcup_i [\mathbb{H}_i]([\mathbf{x}]([t])) \end{aligned}$$

Application: the Car-Trailer



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$



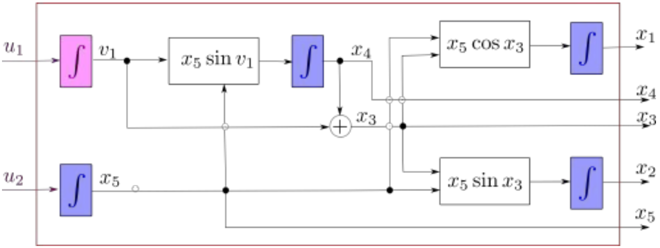
Can we conclude that x_1, x_2, x_3, x_4 belong or not to $\mathscr{R}_0 < u_1, u_2 >$?

Proposition. An integral formulation of the car-trailer is

$$\begin{cases} x_1 &= x_1(0) + \int (x_5 \cos x_3) \\ x_2 &= x_2(0) + \int (x_5 \sin x_3) \\ x_3 &= x_4 + v_1 \\ x_4 &= x_4(0) + \int (x_5 \sin v_1) \\ v_1 &= v_1(0) + \int u_1 \\ x_5 &= x_5(0) + \int u_2 \end{cases}$$

with

$$v_1(0) = x_3(0) - x_4(0).$$



Integral representation of the car-trailer

Proof. Since

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$

We set $v_1 = x_3 - x_4$. We have:

$$\begin{array}{ccc} x_5 & \in & \mathscr{R}_0 < u_2 > \\ v_1 & \in & \mathscr{R}_0 < u_1 > \\ x_4 & \in & \mathscr{R}_0 < v_1, x_5 > \\ x_3 & \in & \mathscr{R}_0 < v_1, x_4 > \\ (x_1, x_2) & \in & \mathscr{R}_0 < x_3, v_5 > \end{array} \quad \Rightarrow \quad (x_1, x_2, x_3, x_4, x_5) \in \mathscr{R}_0 < u_1, u_2 >$$

.

Illustration

Take

$$\mathbf{u}(t) = \mathbf{u}^*(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \text{ and } \mathbf{x}(0) = \mathbf{x}^*(0) = (0, 0, 1, 1.5, 1).$$

Assume that

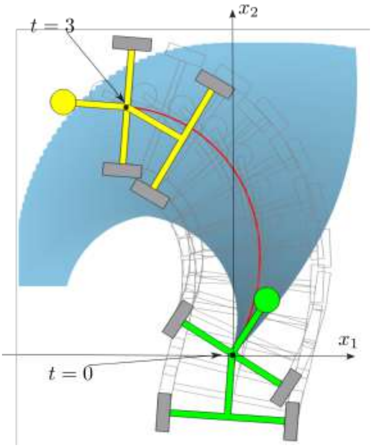
$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \begin{pmatrix} [u_1](t) \\ [u_2](t) \end{pmatrix} = \mathbf{u}^*(t) + 10^{-2} \begin{pmatrix} [-1, 1] \\ [-1, 1] \end{pmatrix}$$

and

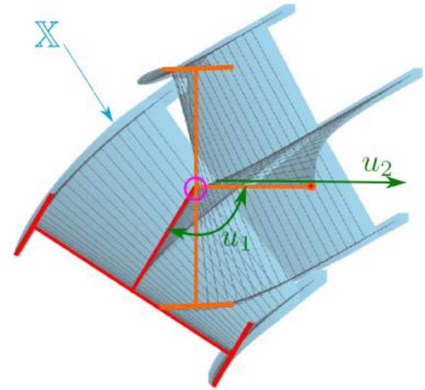
$$\mathbf{x}(0) \in [\mathbf{x}](0) = \mathbf{x}^*(0) + \begin{pmatrix} [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.2, 0.2] \\ [-0.01, 0.01] \\ [-0.001, 000.1] \end{pmatrix}.$$

The interval trajectory in the (x_1, x_2)-space is obtained by:

$$
\begin{array}{rcl}
\text{In:} & & [x_1](0), [x_2](0), [x_3](0), [x_4](0), [x_5](0), [u_1](t), [u_2](t) \\
[v_1](0) & = & [x_3](0) - [x_4](0). \\
[v_1](t) & = & [v_1](0) + \int_0^t [u_1](\tau) d\tau \\
[x_5](t) & = & [x_5](0) + \int_0^t [u_2](\tau) d\tau \\
[x_4](t) & = & [x_4](0) + \int_0^t [x_5](\tau) \cdot \cos([v_1](\tau)) \cdot d\tau \\
[x_3](t) & = & [x_4](t) + [v_1](t) \\
[x_1](t) & = & [x_1](0) + \int_0^t [x_5](\tau) \cdot \cos([x_3](\tau)) \cdot d\tau \\
[x_2](t) & = & [x_2](0) + \int_0^t [x_5](\tau) \cdot \sin([x_3](\tau)) \cdot d\tau
\end{array}
$$



Integral simulation of the car-trailer



$$\mathbb{X} = \bigcup_{t\in[0,t_{\max}]} \bigcup_i \mathbb{H}_i(\mathbf{x}(t)) \subset \bigcup_{[t_k]} \bigcup_i [\mathbb{H}_i]([\mathbf{x}]([t_k]))$$

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